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Towards a Constructive and Predicative Integration Theory of Locally Compact Metric Spaces

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Abstract

Bishop style constructive integration theory constitutes an important milestone in constructive mathematics as it demonstrates the actual feasibility of developing a rich theory of integration within the constructive framework. However, Bishop's approach has the fundamental flaw that it allows impredicativity in the sense that it uses statements that contain quantification over the whole universe of sets. In this thesis, I work towards amending Bishop's theory in order to remove this impredicativity. Furthermore, I try to increase clarity through the explicit use of moduli. First, I introduce the necessary fundamental notions of Bishop Set Theory as presented by Petrakis. Following Bishop's book, I develop the theory of locally compact metric spaces. Lastly, I introduce a notion of integration on locally compact metric spaces and prove that the set of partial functions with compact support constitute an integration space in a sensible manner.

Acknowledgments

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Contents

Introduction

The vast majority of contemporary mathematics is based on classical mathematics (CLASS). The reason for this is that CLASS does one thing very well: It gets the job done. It manages to provide easy-to-use tools to describe highly abstract concepts and therefore caters to the needs of the common mathematician who wants to focus on their respective work rather than the low-level details. Unfortunately, in CLASS those low-level details are rather problematic, as its largest selling point is at the same time also its fundamental flaw: The law of the excluded middle (LEM). At the very center of CLASS, LEM is the axiom that for every statement A it is true that

$$A \vee \neg A$$

or in other words that a statement is either true or false. It is due to this seemingly intuitive assumption that CLASS has lost most of its computational meaning. CLASS allows us to make statements about objects that are inherently inconstructible and what is more, it allows for the unnecessary introduction of non-constructivist methods when an equally valid constructive method would be available.

In the very first chapter, Bishop and Bridges [1] provide a good example that illustrates this non-constructivity in a very accessible manner. In CLASS the defining characteristic of the real numbers is the least-upper-bound-property, the statement that every subset of the real numbers that is bounded from above has a supremum. At first glance this statement seems docile, but its unruliness is revealed when we take a look at a concrete example of such a subset: Let $P(n)$ be some property of a natural number n and consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined as follows:

$$x_n := \begin{cases} 0 & \text{if } P(n) \text{ holds} \\ 1 & \text{else} \end{cases}$$

Now consider the set $\{x_n \mid n \in \mathbb{N}\}$ of the values of $(x_n)_{n\in\mathbb{N}}$, that, being a subset of $\{0, 1\}$, is obviously bounded from above. The least-upper-bound-property now states that this set has a supremum. It does not, however, provide an algorithm or any other means of computing this supremum. This becomes clear if we let $P(n)$ be the statement that n satisfies e. g. the property of the Goldbach conjecture. If the least-upper-bound-property provided such a computational means, we could use it to immediately prove or refute the Goldbach conjecture, depending on whether the computation of the supremum results in 0 or 1.

From a constructivist mathematician's viewpoint, non-constructivist statements about existence are inherently pointless. This includes the least-upper-bound-property discussed above but it actually extends to most of contemporary mathematics. In this sense, non-constructivism makes mathematics philosophically vulnerable.

However, cleansing the whole body of modern mathematics of non-constructivist methods is not an easy feat and transcends the mere removal of LEM. Unfortunately, many definitions and properties of CLASS are aligned with the non-constructivism and are either not true or not sensible in constructivist mathematics. It seems therefore, that abandoning our current framework altogether and building a new foundation might be the solution to the problem rather than only switching to an intuitionistic framework. Historically, there have been different approaches to this matter, depending on how much of the tried concepts people were willing to sacrifice, with maybe intensional type theory being at the other end of the spectrum. As every one of those strategies has its advantages and drawbacks, there is no way to tell which one is the best. According to Bishop, due to the heavy reliance on set-theoretic methods

> [a]ny constructive approach to mathematics will find a crucial test in its ability to assimilate the intricate body of mathematical thought called measure theory, or the theory of integration [1, pp. 215-216].

Needless to say, the focus of the approach of Bishop [2] and later Bishop and Bridges [1] lies in analysis and integration theory. It is a rather informal theory that retains many elements of classical set theory, one of the most significant differences being the reliance on abstract equality relations as an inherent part of sets, rather than only using the definitional equality. Accordingly, the notion of a subset is not limited to extensional subsets with every element actually being definitionally equal to some element of the superset. Another difference is the shift to using function-based definitions instead of set-based ones. This is justified by the fact that functions have proven to be more useful in the constructivist framework as "functions are sharply defined, whereas most sets are fuzzy around the edges" [1, p. 77]. As such, a functional approach similar to the Daniell Integral was chosen over the measure theoretic approach for the integration theory.

The approach of Bishop and Bridges [1] is, however, not completely flawless. One point of criticism is that it is impredicative in the sense that it allows membership properties of sets that contain quantifications over the whole universe. The problem here is that in order to verify if an object satisfies such a property, one would have to check said property for every single set beforehand. In this thesis, we use an approach inspired by the definitions in Petrakis [3] and Petrakis [4] in order to try to reformulate the construction of the integral of Bishop and Bridges [1] in a way that avoids impredicativity. Additionally we try to make definitions and propositions clearer through the explicit use of moduli.

In the first chapter, we establish the fundamental notions of Bishop's set theory as well as constructive real analysis, the building blocks for the later chapters. In the second chapter, we introduce and elaborate on the topic of metric spaces, working towards the definition of local compactness. The third chapter deals with the actual integration theory. Some other important definitions and propositions lead us then to the main result of this thesis stating that the set of continuous functions with compact support constitutes an integration space within our framework.

1 Basic Definitions

In this chapter we introduce the fundamental concepts are required later in the thesis. In the first section, we introduce the notions that lie at the very foundation of Bishop Set Theory. This includes definitions such as *sets*, *functions*, *subsets* and *partial functions*. We also touch the topics of *equality* and *apartness* as well as the *union* and *intersection* of *subsets*. The second section deals with *families* and *sets* of subsets and partial functions. In the third section, we define important notions of constructive real analysis and prove some useful statements. The definitions in the first two sections are adopted from Petrakis [3], while the third section exclusively references Bishop and Bridges [1].

1.1 Basic Notions of Bishop Set Theory

Bishop Set Theory (BST) is a constructive set theory that uses the framework of first-order intuitionistic logic with equality [3]. The two core concepts are *sets* and *functions*, both of which are considered atomic. This is as opposed to other set theories like ZF, where only the notion of set is atomic and functions are modeled as special sets. It is worth noting though, that sets and functions are themselves special cases of so-called totalities and operations which are roughly the equivalent of classes in ZF and are generally avoided.

The primitive (logical) equality of BST is the equality relation $:=$ embodying the fact that two objects are fundamentally the same, rather than the same with respect to a certain non-trivial equality relation. Therefore, if P is some property, then in any case it holds that $[a := b \wedge P(a)] \implies P(b)$.

Within BST we consider the *set of natural numbers* $\mathbb{N}$ as a special primitive set. Its property is that it obeys the Peano axioms and especially the induction property.

Definition 1. A *(defined) totality* X is the structure induced by formula $\mathscr{M}_X$ of first order intuitionistic logic for which we define that $x \in X :\iff \mathscr{M}_X(x)$. In this case we call $\mathscr{M}_X$ the *membership condition* of X. Two totalities X, Y, defined by the membership conditions $\mathscr{M}_X$ and $\mathscr{M}_Y$ respectively, are said to be *definitionally equal*, in symbols $X := Y$, iff for all x it is true that $\mathscr{M}_X(x) :\iff \mathscr{M}_Y(x)$.

A *set* is a pair $(X, =_X)$ satisfying the following properties:

(i) The membership condition $\mathscr{M}_X$ of X is a finite construction. (see remark 1)

(ii) $=_X$ is an equivalence relation, i. e. it is reflexive ($x =_X x$), symmetric ($x =_X y \implies y =_X x$) and transitive ($(x =_X y \wedge y =_X z) \implies x =_X z$).

If x_0 is such that $x_0 \in X$, then say that $(X, =_X)$ is *inhabited* by x_0 or simply an *inhabited set*.

As an auxiliary structure we also define the *universe* $\mathbb{V}_0$ to be a special *open ended* totality that has the special property that it contains every set. The universe $\mathbb{V}_0$ should generally be avoided and we only ever use it in the context of introducing other sets by saying they are a member of $\mathbb{V}_0$. We also use it later to introduce the equality $=_{\mathbb{V}_0}$ of sets.

Remark 1. By *finite construction* we mean that the membership condition is predicative, i. e. that every quantification is a quantification over a proper set, and specifically that there are no quantifications over the whole universe $\mathbb{V}_0$.

Definition 2. Let $(X, =_X), (Y, =_Y)$ be sets. Then we define the *cartesian product* $X \times Y$ in the obvious way as the totality given by the membership condition

$$\mathscr{M}_{X \times Y}(z) :\iff \exists_{x \in X} \exists_{y \in Y} (z := (x, y))$$

and the equality given by

$$(x, y) =_{X \times Y} (x', y') :\iff x =_X x' \wedge y =_Y y'$$

Since there is no quantification over $\mathbb{V}_0$, $X \times Y$ is considered to be a set. If $N \in \mathbb{N}$ and $(X_0, =_{X_0}), \dots, (X_N, =_{X_N})$ are sets, then we similarly define the cartesian product

$$\prod_{j=0}^{n} X_j$$

Definition 3. Let $(X, =_X)$ be a set. A relation $\neq_X$ is called an *apartness relation* on X if for all $x, y \in X$ the following hold:

(i) $(x =_X y \wedge x \neq_X y) \implies \bot$

(ii) $x \neq_X y \implies y \neq_X x$

(iii) $x \neq_X y \implies \forall_{z \in X}(z \neq_X x \vee z \neq_X y)$

In this case we call the triple $(X, =_X, \neq_X)$ a *set with apartness.* [3, p. 11]

Definition 4. For totalities X and Y, a *non-dependent assignment routine* f from X to Y, denoted by $f : X \rightsquigarrow Y$, is a finite routine that assigns an element $y \in Y$ to each given element x of X. In this case we also write $f(x) := y$. Two non-dependent assignment routines $f, g : X \rightsquigarrow Y$ are definitionally equal iff they map definitionally equal elements to definitionally equal elements, i. e.

$$f := g :\Longleftrightarrow \forall_{x \in X}(f(x) := g(x))$$

If X, Y and Z are totalities and $f : X \rightsquigarrow Y$ as well as $g : Y \rightsquigarrow Z$ are non-dependent assignment routines, then we define the *composition* $g \circ f$ by

$$(g \circ f)(x) := g(f(x))$$

If $(X, =_X)$ and $(Y, =_Y)$ are in fact sets, a non-dependent assignment routine $f : X \rightsquigarrow Y$ is called an *operation.* We define the set $\mathcal{O}(X, Y)$ of all operations from X to Y together with the equality $=_{\mathcal{O}(X,Y)}$ given by

$$f =_{\mathcal{O}(X,Y)} g :\Longleftrightarrow \forall x \in X(f(x) =_Y g(X))$$

If $f \in \mathcal{O}(X, Y)$ respects the equalities $=_X$ and $=_Y$, i. e.

$$\forall_{x,x' \in X}(x =_X x' \implies f(x) =_Y f(x'))$$

then f is called a *function* and we write $f : X \to Y$. Similarly to $\mathcal{O}(X, Y)$ we define the set $\mathcal{F}(X, Y)$ to be the set of all functions from X to Y. It is equipped with the equality $=_{\mathcal{F}(X,Y)}$ derived from the equality $=_{\mathcal{O}(X,Y)}$.

If X and Y are each equipped with apartness relations $\neq_X$ and $\neq_Y$ respectively, $f : X \to Y$ is said to be *strongly extensional* iff it respects inequalities in the obvious way, i. e.

$$\forall_{x,x' \in X}(f(x) \neq_Y f(x') \implies x \neq_X x')$$

In the case that X and Y are equipped wiht the apartness relations, we define the apartness $\neq_{\mathcal{O}(X,Y)}$ by letting

$$f \neq_{\mathcal{O}(X,Y)} g :\Longleftrightarrow \exists x \in X(f(x) \neq_Y g(x))$$

Similarly, $\neq_{\mathcal{F}(X,Y)}$ is derived from $\neq_{\mathcal{O}(X,Y)}$.

Lastly, $f : X \to Y$ is said to be an *embedding* iff

$$\forall_{x,x' \in X}(f(x) =_Y f(x') \implies x =_X x')$$

and we write $f : X \hookrightarrow Y$. We also define the Set $\mathrm{Emb}(X, Y)$ to be the set of all embeddings from X to Y. Its equality $=_{\mathrm{Emb}(X,Y)}$ is derived from $\mathcal{O}(X, Y)$.

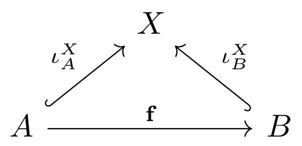
Definition 5. For a set I and a non-dependent assignment routine $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, a *dependent operation* Φ over λ_0 is itself an assignment routine, where for every $i \in I$ we have that $\Phi_i := \Phi(i) \in \lambda_0(i)$. We sometimes also write

$$\Phi : \bigcurlywedge_{i \in I} \lambda_0(i)$$

instead. Two non-dependent assignment routines $\Phi, \Psi : \bigcurlywedge_{i\in I} \lambda_0(i)$ are considered equal iff for every $i \in I$ we have that $\Phi(i) =_{\lambda_0(i)} \Psi(i)$.

Definition 6. Let $(X, =_X)$ and $(A, =_A)$ be sets. If $\iota_A^X : A \hookrightarrow X$ is an embedding, the pair (A, ι_A^X) is called a *subset*. In the case that $(A, =_A)$ is actually inhabited by a_0, we call (A, ι_A^X, a_0) an *inhabited subset*.

If (B, ι_B^X) is another subset of X, then we say that (A, ι_A^X) is a *subset* of (B, ι_B^X), in symbols $(A, \iota_A^X) \subseteq (B, \iota_B^X)$, iff there is a modulus of the subset property, i. e. there exists a function $\mathbf{f} : A \to B$ such that the following diagram commutes with respect to the equalities $=_X$ and $=_Y$:



In this case we write $\mathbf{f} : A \subseteq B$. We denote by $\mathscr{P}(X)$ the totality of all subsets of X. Its equality is given by the condition

$$(A, \iota_A^X) =_{\mathcal{P}(X)} (B, \iota_B^X) :\iff A \subseteq B \wedge B \subseteq A$$

In this case, if $\mathbf{f} : A \subseteq B$ and $\mathbf{g} : B \subseteq A$, then we write $(f, g) : A =_{\mathcal{P}(X)} B$.

Remark 2. The powerset $\mathscr{P}(X)$ of a set X is not considered to be a set, since the membership condition contains a quantification over the universe:

$$\mathscr{M}_{\mathcal{P}(X)}(x) :\iff \exists_{A \in \mathbb{V}_0} \exists_{\iota_A^X \in \mathcal{F}(A,X)}(x := (A, \iota_A^X))$$

Definition 7. Let $(X, =_X)$ be a set and P a property given by a formula $P(x)$. We call $P(x)$ an *extensional property* on X iff for every $x, y \in X$ it holds that

$$(x =_X y \wedge P(x)) \implies P(y)$$

The extensional property $P(x)$ induces a canonical subset X_P of X. The membership condition $\mathscr{M}_{X_P}$ is given by

$$\mathscr{M}_{X_P}(x) :\iff \mathscr{M}_X(x) \wedge P(x)$$

and the equality $=_{X_P}$ is defined in the obvious way by letting

$$x =_{X_P} y :\iff x =_X y$$

In this case we call the subset (X_P, id) of X the *extensional subset* of X defined by P. We also write $\{x \in X \mid P(x)\}$ instead of X_P.

Remark 3. Let $(X, =_X)$ be a set and consider the set $X \times X$. A useful example for an extensional subset is $D(X)$, the *diagonal of* X. It is defined as

$$D(X) := \{(x, y) \in X \times X \mid x =_X y\}$$

Another useful extensional subset is the image of a function. Let $(X, =_X)$ and $(Y, =_Y)$ be sets and $f : X \to Y$ a function. Then we define the set

$$f(X) := \{y \in Y \mid \exists x \in X\; f(x) =_Y y\}$$

together with a modulus $\mathbf{f}^{-1} : f(X) \to X$ that gives us a preimage for each element of $f(X)$. Note that $\mathbf{f}^{-1}$ should not be confused with the inverse function f^{-1} that only exists iff f is a bijection.

Definition 8. Let (A, ι_A^X) and (B, ι_B^X) be subsets of X. The *union* $A \cup B$ is the subset of X defined by

$$\mathscr{M}_{A \cup B}(x) :\iff \mathscr{M}_A(x) \vee \mathscr{M}_B(x)$$

Its embedding is given by

$$\iota_{A \cup B}^X(z) := \begin{cases} \iota_A^X(z) & z \in A \\ \iota_B^X(z) & z \in B \end{cases}$$

and the equality is given by the formula

$$z =_{A\cup B} w :\iff \iota^X_{A\cup B}(z) =_X \iota^X_{A\cup B}(w)$$

The *intersection* $A \cap B$ is defined as the set

$$A \cap B := \{(a, b) \in A \times B \mid \iota^X_A(a) =_X \iota^X_B(b)\}$$

of $A \times B$ together with the embedding given by $\iota^X_{A\cap B}(a, b) := \iota^X_A(a)$.

A partial function is a function that is defined on a subset rather than the whole set. Unfortunately the totality of partial functions is not a set as the membership condition, like for the powerset, requires the quantification over the universe $\mathbb{V}_0$.

Definition 9. Let X, Y be sets, (A, ι^X_A) a subset of X and $f^Y_A : A \to Y$. Then we call (A, ι^X_A, f^Y_A) a *partial function* from X to Y and write $f^Y_A : X \rightharpoonup Y$. In addition, we define the totality $\mathfrak{F}(X, Y)$ of partial functions from X to Y. Two partial functions are considered equal, i. e. $(A, \iota^X_A, f^Y_A) =_{\mathfrak{F}(X,Y)} (B, \iota^X_B, f^Y_B)$, if there are moduli $\mathbf{e}_{AB} : A \to B$ and $\mathbf{e}_{BA} : B \to A$ such that

$$\begin{aligned} \iota^X_A &=_{\mathcal{F}(A,X)} \iota^X_B \circ \mathbf{e}_{AB} & \qquad \iota^X_B &=_{\mathcal{F}(B,X)} \iota^X_A \circ \mathbf{e}_{BA} \\ f^Y_A &=_{\mathcal{F}(A,Y)} f^Y_B \circ \mathbf{e}_{AB} & \qquad f^Y_B &=_{\mathcal{F}(B,Y)} f^Y_A \circ \mathbf{e}_{BA} \end{aligned}$$

i. e. in the corresponding diagram of figure 1.1 the upper and lower triangles each commute:

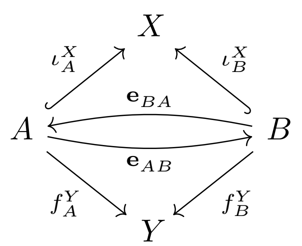


Figure 1.1: equality of partial functions

1.2 Families and Sets of Subsets and Partial Functions

In the following chapter we define the notions of families and sets of subsets as well as partial functions in accordance to the definitions found in Petrakis [3]. They are essential in our discussions as they allow us to avoid the impredicativity that arises from the use of the powerset in the usual definition of sets of subsets and sets of partial functions.

Definition 10. Let $(X, =_X)$ and $(I, =_I)$ be sets. Let further $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$ be a non-dependent assignment routine, $\mathscr{E}^X : \bigcurlywedge_{i \in I} \mathscr{F}(\lambda_0(i), X)$ a dependent operation where for every $i \in I$ we have that $\mathscr{E}^X(i)$ is an embedding and

$$\lambda_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathscr{F}(\lambda_0(i), \lambda_0(j))$$

a dependent operation where $\lambda_1(i, i) := \mathrm{id}_{\lambda_0(i)}$ for every $i \in I$. Then we call $(\lambda_0, \mathscr{E}^X, \lambda_1)$ an *I-family of subsets* of X iff for every $i, j \in I$ such that $i =_I j$ we have that

$$\mathscr{E}^X(i) =_{\mathcal{F}(\lambda_0(i), X)} \mathscr{E}^X(j) \circ \lambda_1(i, j)$$
$$\mathscr{E}^X(j) =_{\mathcal{F}(\lambda_0(j), X)} \mathscr{E}^X(i) \circ \lambda_1(j, i)$$

i. e. if the the diagrams in figure 1.2 commute.

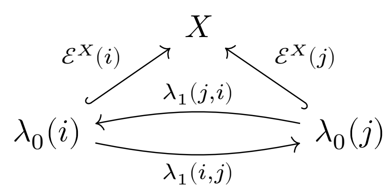


Figure 1.2: family of subsets

Remark 4. In other words, the I-family $(\lambda_0, \mathscr{E}^X, \lambda_1)$ of subsets of X consists of the indexing set I as well as the functions λ_0 and $\mathscr{E}^X$ that specify the indexed subset $(\lambda_0(i), \mathscr{E}^X(i))$ for every $i \in I$. λ_1 ensures that λ_0 respects the equality $=_X$.

Definition 11. Let $(X, =_X), (Y, =_Y)$ and $(I, =_I)$ be sets. Let further $(\lambda_0, \mathscr{E}^X, \lambda_1) \in \mathrm{Fam}(I, X)$ and $\mathscr{P}^Y : \bigcurlywedge_{i \in I} \mathscr{F}(\lambda_0(i), Y)$. We call $(\lambda_0, \mathscr{E}^X, \lambda_1, \mathscr{P}^Y)$

an *I-family of partial functions* from X to Y, iff the following equalities hold for all $i, j \in I$ with $i =_I j$:

$$\mathscr{E}^X(i) =_{\mathcal{F}(\lambda_0(i),X)} \mathscr{E}^X(j) \circ \lambda_1(i,j)$$
$$\mathscr{E}^X(j) =_{\mathcal{F}(\lambda_0(j),X)} \mathscr{E}^X(i) \circ \lambda_1(j,i)$$
$$\mathscr{P}^Y(i) =_{\mathcal{F}(\lambda_0(i),Y)} \mathscr{P}^Y(j) \circ \lambda_1(i,j)$$
$$\mathscr{P}^Y(j) =_{\mathcal{F}(\lambda_0(j),Y)} \mathscr{P}^Y(i) \circ \lambda_1(j,i)$$

i. e. if the diagrams in figure 1.3 commute:

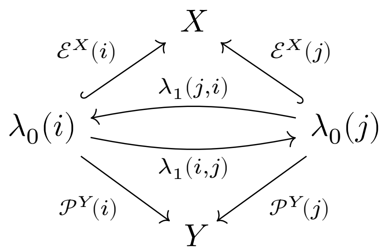


Figure 1.3: family of partial functions

Definition 12. Additionally we define the the totalities $\mathrm{Fam}(I, X)$ and $\mathrm{Fam}(I, X, Y)$ to be the totality of all families of subsests of X and the totality of all families of partial functions from X to Y. For a detailed discussion, please refer to Petrakis [3].

Next we consider the special case for the above definitions where the non-dependent assignment routines λ_0 have the embedding property. In this case we call the structures "sets" instead of "families".

Definition 13. Let $(X, =_X)$ and $(I, =_I)$ be sets and let further $(\lambda_0, \mathscr{E}^X, \lambda_1) \in \mathrm{Fam}(I, X)$. If for all $i, j \in I$ we have that

$$\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j) \implies i =_I j$$

then we call $(\lambda_0, \mathscr{E}^X, \lambda_1)$ an *I-set of subsets* of X. Additionally we define $\mathrm{Set}(I, X)$ to be the totality of I-sets of subsets of X. The equality $=_{\mathrm{Set}(I,X)}$ is given by the equality $=_{\mathrm{Fam}(I,X)}$.

Definition 14. Let $(X,=_X),(Y,=_Y)$ and $(I,=_I)$ be sets and let $(\lambda_0,\mathscr{E}^X,\lambda_1,\mathscr{P}^Y)\in \mathrm{Fam}(I,X,Y)$. We call $(\lambda_0,\mathscr{E}^X,\lambda_1,\mathscr{P}^Y)$ an *I-set of partial functions* from X to Y iff for all $i,j\in I$ we have that

$$(\lambda_0(i),\mathscr{E}^X(i),\mathscr{P}^Y(i)) =_{\mathfrak{F}(X,Y)} (\lambda_0(j),\mathscr{E}^X(j),\mathscr{P}^Y(j))$$

Similar to sets of subsets, we define the totality $\mathrm{Set}(I,X,Y)$ of I-sets of partial functions from X to Y, equipped with the equality $=_{\mathrm{Set}(I,X,Y)}$ given by the equality $=_{\mathrm{Fam}(I,X,Y)}$.

Proposition 1. Let $(X,=_X)$ and $(Y,=_Y)$ be sets and (F,ι_F) be a subset of $\mathscr{F}(X,Y)$. We define:

(i) $\lambda_0 : F \rightsquigarrow \mathbb{V}_0$ is the constant non-dependent assignment routine with $\lambda_0(f) := X$ for all $f\in F$.

(ii) $\mathscr{E}^X : \bigcurlywedge_{f\in F}\mathscr{F}(X,X)$ is the constant dependent operation with $\mathscr{E}^X(f) := \mathrm{id}_X$ for all $f\in F$.

(iii) $\lambda_1 : \bigcurlywedge_{(f,g)\in D(F)}\mathscr{F}(X,X)$ is the constant dependent operation for which $\lambda_1(f,g) := \mathrm{id}_X$ holds for all $f,g\in F$ such that $f =_F g$.

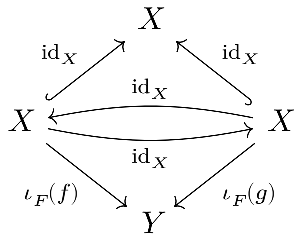
(iv) $\mathscr{P}^Y : \bigcurlywedge_{f\in F}\mathscr{F}(X,Y)$ is the dependent operation given by $\mathscr{P}^Y(f) := \iota_F(f)$ for every $f\in F$

Then $\bar{F} := (\lambda_0,\mathscr{E}^X,\lambda_1,\mathscr{P}^Y)$ is a well-defined F-set of partial functions.

Proof. At first, we see that our definitions are well-defined and match the signatures. The signatures in turn fit the requirements for $\bar{F}$ to be an F-family of partial functions from X to Y, and it remains to show that $\bar{F}$ respects the equality of F. Therefore let $f,g\in F$ such that $f =_F g$. Then due to the extensionality of ι_F, i. e. $\iota_F(f) =_{\mathcal{F}(X,Y)} \iota_F(g)$, the following hold true:

$$\mathscr{E}^X(f) := \mathrm{id}_X =_{\mathcal{F}(X,Y)} \mathrm{id}_X\circ\mathrm{id}_X := \mathscr{E}^X(g)\circ\lambda_1(f,g)$$
$$\mathscr{E}^X(g) := \mathrm{id}_X =_{\mathcal{F}(X,Y)} \mathrm{id}_X\circ\mathrm{id}_X := \mathscr{E}^X(f)\circ\lambda_1(g,f)$$
$$\mathscr{P}^Y(f) := \iota_F(f) =_{\mathcal{F}(X,Y)} \iota_F(g) =_{\mathcal{F}(X,Y)} \iota_F(g)\circ\mathrm{id}_X := \mathscr{P}^Y(g)\circ\lambda_1(f,g)$$
$$\mathscr{P}^Y(g) := \iota_F(g) =_{\mathcal{F}(X,Y)} \iota_F(f) =_{\mathcal{F}(X,Y)} \iota_F(f)\circ\mathrm{id}_X := \mathscr{P}^Y(f)\circ\lambda_1(g,f)$$

I. e. the following diagrams commute:



Together we have that $\bar{F} \in \mathrm{Fam}(I, X, Y)$. To show that $\bar{F}$ is in fact a *set* of partial functions, let $f, g \in F$ and $\mathbf{e}_1, \mathbf{e}_2 : X \to X$ such that

$$(\mathbf{e}_1, \mathbf{e}_2) : (X, \mathrm{id}_X, \iota_F(f)) =_{\mathfrak{F}(X,Y)} (X, \mathrm{id}_X, \iota_F(g))$$

By definition this implies that $\mathrm{id}_X =_{\mathcal{F}(X,X)} \mathrm{id}_X \circ \mathbf{e}_1$ and $\mathrm{id}_X =_{\mathcal{F}(X,X)} \mathrm{id}_X \circ \mathbf{e}_2$, i. e. $\mathbf{e}_1 =_{\mathcal{F}(X,X)} \mathbf{e}_2 =_{\mathcal{F}(X,X)} \mathrm{id}_X$. With the other part of the definition we see that

$$\iota_F(f) =_{\mathcal{F}(X,Y)} \iota_F(g) \circ \mathrm{id}_X =_{\mathcal{F}(X,Y)} \iota_F(g)$$

Since ι_F is an embedding, we see that $f =_F g$, i. e. $\bar{F} \in \mathrm{Set}(I, X, Y)$. □

Last but not least note that there is also a predicative way to talk about intersections and unions of families of subsets. For the exact workings, please refer to Petrakis [3, pp. 91, 97]. For now it suffices to know that if $(X, =_X)$ and $(I, =_I)$ are sets and $(\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathrm{Fam}(I, X)$ is an I-family of subsets of X, then the *intersection* and *union* of $(\lambda_0, \mathcal{E}^X, \lambda_1)$ are denoted by the symbols

$$\bigcap_{i \in I} \lambda_0(i) \qquad \text{and} \qquad \bigcup_{i \in I} \lambda_0(i)$$

respectively and that they are defined predicatively in a sensible manner.

1.3 Constructive Real Analysis

This section briefly introduces some of the fundamental properties of the real numbers as described in chapter 2 of Bishop and Bridges [1]. A detailed discussion as well as corresponding proofs can be found in the original work.

In the following we assume that in addition to the natural numbers $\mathbb{N}$ as well as the positive natural numbers $\mathbb{N}^+$ and for $n \in \mathbb{N}$ the set $\mathbb{N}^{\leq n}$ of all natural numbers smaller or equal than n, the set $\mathbb{Z}$ of integers and the set $\mathbb{Q}$ of rational numbers have already been constructed from the set $\mathbb{N}$ in the usual algebraic manner and are therefore equipped with the usual arithmetic operations as well as the absolute value function. Furthermore, we assume they are equipped with the usual total orderings $<_{\mathbb{N}}$, $<_{\mathbb{Z}}$ and $<_{\mathbb{Q}}$. Note that for each of them the dichotomy

$$x \leq y \vee y < x$$

holds, where $x \leq y :\iff (y < x) \implies \bot$. However, this is not true for the real numbers defined below.

Definition 15. A sequence $(x_n)_{n\in\mathbb{N}^+}$ in the rational numbers is a *real number* or *regular sequence*, iff for all $m, n \in \mathbb{N}^+$ we have that

$$|x_m - x_n| \leq m^{-1} + n^{-1}$$

The set of real numbers is defined as $(\mathbb{R}, =_{\mathbb{R}})$ consisting of the following:

(i) The totality $\mathbb{R}$ of all regular sequences.

(ii) The equality $=_{\mathbb{R}}$ where two real numbers $(x_n)_{n\in\mathbb{N}^+}$ and $(y_n)_{n\in\mathbb{N}^+}$ are considered equal, i. e. $(x_n)_{n\in\mathbb{N}^+} =_{\mathbb{R}} (y_n)_{n\in\mathbb{N}^+}$, iff for all $n \in \mathbb{N}^+$ we have that $|x_n - y_n| \leq 2^{n-1}$. $=_{\mathbb{R}}$ constitutes an equivalence relation. For the sake of convenience, we refer to $=_{\mathbb{R}}$ simply as $=$.

If X is a set, then for the sake of convenience we define $\mathscr{F}(X) := \mathscr{F}(X, \mathbb{R})$.

For a real number $x := (x_n)_{n\in\mathbb{N}^+}$ we define the *canonical bound* $\mathscr{K}\_x \in \mathbb{N}^+$ for x to be the least natural number such that $|x_1| + 2 < \mathscr{K}\_x$. In this case, for all $(y_n)_{n\in\mathbb{N}^+} \in \mathbb{R}$ such that $(x_n)_{n\in\mathbb{N}^+} = (y_n)_{n\in\mathbb{N}^+}$, it holds that $|y_n| < K_x$ for every $n \in \mathbb{N}^+$.

Remark 5. The operation $\mathscr{K}\_ : \mathbb{R} \to \mathbb{N}^+$ is an example of an operation that is not a function. Both $x := (0, 1, 1, \dots)$ and $y := (2, 1, 1, \dots)$ are real numbers representing the number 1, i. e. $x = y = 1$, however, it is $\mathscr{K}\_x := 2$ and $\mathscr{K}\_y := 4$.

Definition 16. Let $(x_n)_{n\in\mathbb{N}^+} \in \mathbb{R}$. We say that $(x_n)_{n\in\mathbb{N}^+}$ is *positive*, iff there exists $n \in \mathbb{N}^+$ such that

$$n^{-1} < x_n$$

It is called *non-negative* iff for all $n \in \mathbb{N}^+$ we have that

$$-n^{-1} \leq x_n$$

Now we can easily define the sets $\mathbb{R}^+$ of all the positive real numbers as well as the set $\mathbb{R}^{\geq 0}$ of all non-negative real numbers. For $\mathbb{R}^+$, however an object $x \in \mathbb{R}^+$ is always accompanied by a specific *modulus of positivity* $\mathbf{n} \in \mathbb{N}^+$.

Additionally, we define a real number $x \in \mathbb{R}$ to be *negative*, iff $-x \in \mathbb{R}^+$, and *non-positive* if $-x \in \mathbb{R}^{\geq 0}$.

Definition 17. Let $x, y \in \mathbb{R}$. We define the order relations $<, >, \leq, \geq$ as follows:

$$x < y :\Longleftrightarrow\ y - x \in \mathbb{R}^+$$
$$x \leq y :\Longleftrightarrow\ y - x \in \mathbb{R}^{\geq 0}$$

Furthermore, we say that $y > x$ iff $x < y$ and $y \geq x$ iff $x \leq y$. We define the real *intervals* as extensional subsets in the usual way.

Finally, we say that $x \neq_{\mathbb{R}} y$ or conveniently $x \neq y$, iff $x < y \vee y < x$.

Lemma 1. If $x, y \in \mathbb{R}$ such that $x < y$, then there exists $\alpha \in \mathbb{R}$ such that $x < \alpha < y$.

Corollary 1. Let $x, y \in \mathbb{R}$ such that $x < y$. Then the *constructive dichotomy* holds, i. e. for every $z \in \mathbb{R}$ we have that $x < z \vee z < y$.

Lemma 2. Let $x, y \in \mathbb{R}$. If $x > y$ implies $0 = 1$, i. e. $\neg(x > y)$, then it holds that $x \leq y$.

Definition 18. Let (A, ι) be a subset of $\mathbb{R}$ inhabited by $a_0 \in A$. Then we say that A is *bounded from above (resp. below)* iff there is $b \in \mathbb{R}$ such that $\iota(a) \leq b$ (resp. $b \leq \iota(a)$) for every $a \in A$. b is said to be the *supremum (resp. infimum)* of A iff for every $\varepsilon \in \mathbb{R}^+$ there is some $a \in A$ such that $b < a + \varepsilon$ (resp. $a - \varepsilon < b$).

If they exist, the supremum and infimum are unique. We therefore write $b = \sup A$ (resp. $b = \inf A$).

Definition 19. Let $x, y \in \mathbb{R}$. We also define the *minimum* min as well as the *maximum* max of x and y in the usual way as one of the two elements such that

$$x, y \leq \max(x, y) \qquad \text{and} \qquad \min(x, y) \leq x, y$$

Additionally, if $(X, =_X)$ is a set and $f, g \in \mathscr{F}(X)$, we define the functions $f \wedge g, f \vee g \in \mathscr{F}(X)$ such that

$$(f \vee g)(x) := \max(x, y) \qquad (f \wedge g)(x) := \min(x, y)$$

Remark 6. In constructive mathematics, supremum and infimum (like many other constructive concepts) are stronger concepts than their counterparts in classical mathematics as their existence requires an actual construction.

The fundamental theorem in classical analysis that every nonvoid subset of $\mathbb{R}$ that is bounded from above has a supremum is not valid in constructive analysis. There is, however, a constructive counterpart for this least-upper-bound-property in the form of proposition 2.

Proposition 2. Let (A, ι) be a subset of $\mathbb{R}$ inhabited by x_0 that is bounded from above, i. e. there is $b \in \mathbb{R}$ such for all $a \in A$ it holds that $\iota(a) < b$. Then $\sup A$ exists iff for all $x, y \in \mathbb{R}$ such that $x < y$ we have that one of the following cases holds:

(i) y is an upper bound of A

(ii) there is some $a \in A$ such that $x < \iota(a)$

2 Metric Spaces

In this chapter, we establish the necessary definitions to later develop our theory of *locally compact metric spaces*. In the first section we introduce the notion of *metric spaces*, *locatedness* as well as *uniform continuity*. This is followed by sections covering *boundedness*, *completeness*, *total boundedness* and *compactness*. In the sixth section we then define *local compactness* as well as *(Bishop) continuity* and prove some practical statements.

At large, we follow the outline as well as the proofs presented in the chapter about metric spaces of Bishop and Bridges [1], amending most of the definitions in the way suggested by Petrakis [4] in order to avoid impredicativity. The two exceptions of this are the third section about completeness and the fifth section about compactness that exclusively reference the respective sections of Bishop and Bridges [1].

2.1 Basic Definitions

A metric space is essentially a set equipped with a notion of distance between its elements. The notion of metric spaces is an essential part of constructive analysis. In this section we introduce its exact definition as well as some basic related concepts. The definitions are adopted from Petrakis [4].

Definition 20. Let $(X, =_X)$ be a set and $d : X \times X \to \mathbb{R}^{\geq 0}$ a function. d is a *metric* iff each of the following properties is satisfied for all $x, y, z \in X$:

(i) $d(x, y) = 0 \iff x =_X y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$

In this case, we call (X, d) a *metric space*.

Example. For every $n \in \mathbb{N}$ $\mathbb{R}^n$ equipped with the standard euclidean distance, i. e. $d_\epsilon\left(\sum_{i=1}^n x_i e_i, \sum_{i=1}^n y_i e_i\right) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$, constitutes a metric space.

Definition 21. Let $(X, =_X, d)$ be a metric space. Let further (A, ι) be a subset of X. (A, ι) can also be viewed as a proper metric space with the induced metric d_A, given by

$$d_A(x, y) := d(\iota(x), \iota(y))$$

for all $x, y \in A$. In this case we also call (A, ι) a *metric subspace* of X. If A is an extensional subset of X, we have that $d_A(x, y) := d(x, y)$ and therefore we often simply use d for the sake of simplicity.

Definition 22. Let $(X, =_X, d)$ be a metric space. d induces an apartness relation on X by letting $x \neq_X y \iff d(x, y) >_{\mathbb{R}} 0$ for every $x, y \in X$. We call $\neq_X$ the *canonical apartness of X*.

Definition 23. Let $(X, =_X, d)$ be a metric space, $x_0 \in X$ and $R \in \mathbb{R}^+$. The *open* and *closed ball* of radius R about x_0 are the extensional subsets defined by

$$[d_{x_0} < R] := \{x \in X \mid d(x, x_0) < R\}$$
$$[d_{x_0} \leq R] := \{x \in X \mid d(x, x_0) \leq R\}$$

Definition 24. Let $n \in \mathbb{N}$ and $(X_1, d_1), \dots, (X_n, d_n)$ be a finite sequence of metric spaces. We define the metric d on the *cartesian product* $X := \prod_{i=1}^{n} X_i$ by letting

$$d(x, y) := \sum_{i=1}^{n} d_i(x_i, y_i)$$

for every $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X$. Then (X, d) is a metric space.

Definition 25. Let $(X, =_X, d_X)$ and $(Y, =_Y, d_Y)$ be metric spaces and $f : X \to Y$ a function. Additionally let $\boldsymbol{\omega} : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. f is said to be *uniformly continuous* with *modulus of uniform continuity* $\boldsymbol{\omega}$ iff for all $x, y \in X$ and $\varepsilon \in \mathbb{R}^+$ we have that

$$d_X(x, y) \leq \boldsymbol{\omega}(\varepsilon) \implies d_Y(f(x), f(y)) \leq \varepsilon$$

Definition 26. Let (X, d) be a metric space, $(x_n)_{n \in \mathbb{N}}$ a sequence of elements of X, $x \in X$ and $\mathbf{N} : \mathbb{R}^+ \to \mathbb{N}$. We say that $(x_n)_{n \in \mathbb{N}}$ *converges* towards its *limit* x iff for every $\varepsilon \in \mathbb{R}^+$ and every $n \in \mathbb{N}$ with $n > \mathbf{N}(\varepsilon)$ we have that $d(x_n, x) \leq \varepsilon$. In this case we also use the notation $\lim_{n \to \infty} x_n = x$.

Definition 27. Let $(X, =_X, d)$ be a metric space and (A, ι) a subset. We say that (A, ι) is *located* iff the following distance exists for every $x \in X$:

$$d(x, A) := \inf \{d(x, \iota(y)) \mid y \in A\}$$

2.2 Boundedness

A bounded subset has the special property, that the distance between two elements is bounded from above by some real number. We model this by requiring the subset to be inhabited by some element, and by giving a bound on the distance an arbitrary element is allowed to have from this inhabiting element. The definitions in this chapter are adopted from Petrakis [4].

Definition 28. Let $(X, =_X, d)$ be a metric space inhabited by x_0 and $\mathbf{M} \in \mathbb{R}^+$. X is *bounded* with *modulus of boundedness* $\mathbf{M}$ and x_0, iff for all $x \in X$ we have that $d(x_0, x) \leq M$. If $\operatorname{diam} X := \sup \{d(x, y) \mid x, y \in X\}$ exists, we call $\operatorname{diam} X$ the *diameter* of X.

If (A, ι) is a metric subspace of X inhabited by a_0, we say that A is a bounded metric subset of X iff there is some $\mathbf{M}_A \in \mathbb{R}^+$ such that $(A, =_A, d_A)$ is a *bounded* with *modulus of boundedness* $\mathbf{M}_A$ and a_0.

Proposition 3. Let (X, d) be a metric space and (A, ι) a bounded subset with modulus of boundedness $\mathbf{M} \in \mathbb{R}^+$ and $a_0 \in A$. For every $a \in A$ the subset (A, ι) is also bounded with modulus of boundedness $\mathbf{M} + d(a_0, a)$ and a.

Proof. Let $x \in A$. We have that

$$d(x, a) \leq d(x, a_0) + d(a_0, a) \leq \mathbf{M} + d(a_0, a)$$

. □

Lemma 3. Let (X, d) be a metric space inhabited by some x_0. For every $n \in \mathbb{N}^+$ the extensional subset $[d_{x_0} \leq n]$ is bounded with modulus of boundedness n and x_0.

Proof. This immediately follows from the fact that $d(x_0, x) \leq n$ for all $x \in [d_{x_0} \leq n]$ by definition. □

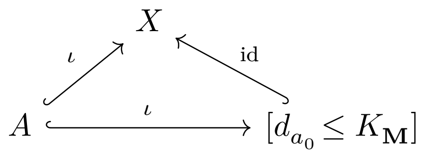
Proposition 4. Let (X, d) be a metric space and (A, ι) a bounded metric subset with modulus of boundedness $M \in \mathbb{R}^+$ and a_0. Then A is contained in a closed ball about $\iota(a_0)$, i.e. there exists $n \in \mathbb{N}^+$ such that $(A, \iota) \subseteq [d_{\iota(a_0)} \leq$

$n]$. In addition, if (Y, d_Y) is another metric space, then for any uniformly continuous function $f : [d_{\iota(a_0)} \leq n] \to Y$ with modulus of uniform continuity $\boldsymbol{\omega}$, the restriction function $f\,|_A \colon A \to Y$ defined by $f\,|_A\,(x) := f(\iota(x))$ is also uniformly continuous with modulus of uniform continuity $\boldsymbol{\omega}$.

Proof. If $\mathscr{K}$_**M** denotes the canonical bound of **M**, the subset property

$$(A, \iota) \subseteq [d_{\iota(a_0)} \leq K_{\mathbf{M}}]$$

is naturally realised by the embedding ι as for every $a \in A$ we have that $d\,(\iota(a_0), \iota(a)) = d\,(a_0, a) \leq M \leq K_{\mathbf{M}}$.



Now let $f : [d_{a_0} \leq K_{\mathbf{M}}] \to Y$ be uniformly continuous with modulus of uniform continuity $\boldsymbol{\omega}$ and let $\varepsilon > 0$ as well as $x, y \in A$ such that $d_A\,(x, y) < \boldsymbol{\omega}_\varepsilon$. Then by definition of d_A, we have that

$$d\,(\iota(x), \iota(y)) = d_A\,(x, y) < \boldsymbol{\omega}_\varepsilon$$

as well and therefore it follows that

$$d_Y(f\,|_A\,(x), f\,|_A\,(y)) = d_Y(f(\iota(x)), f(\iota(y))) \leq \varepsilon$$

by definition of $\boldsymbol{\omega}$. Thus $\boldsymbol{\omega}$ is also a modulus of uniform continuity for $f\,|_A$. □

2.3 Completeness

In the following section we generalize the construction of the set of real numbers $\mathbb{R}$ to arbitrary metric spaces by means of Cauchy sequences. We therefore briefly discuss Cauchy sequences and other related definitions and then define the notion of completeness of a metric space as introduced in chapter 4 of Bishop and Bridges [1]. For a more detailed discussion as well as proofs, please refer to the original work.

Definition 29. Let (X, d) be a metric space and $(x_n)_{n \in \mathbb{N}^+}$ a sequence of elements of X. We call $(x_n)_{n \in \mathbb{N}^+}$ *regular* iff for every $m, n \in \mathbb{N}^+$ the following holds:

$$d\,(x_m, x_n) \leq m^{-1} + n^{-1}$$

We then define the set $\tilde{X}$ of all regular sequences of X and call it the *completion* of X*completion*. Its equality is given by

$$(x_n)_{n\in\mathbb{N}^+} =_{\tilde{X}} (y_n)_{n\in\mathbb{N}^+} :\iff d(x_n, y_n) \leq 2n^{-1}$$

The completion $\tilde{X}$ is itself a metric space, equipped with the metric $\tilde{d}$ well-defined by

$$\tilde{d}\left((x_n)_{n\in\mathbb{N}^+}, (y_n)_{n\in\mathbb{N}^+}\right) := \lim_{n\to\infty} d(x_n, y_n)$$

for all $(x_n)_{n\in\mathbb{N}^+}, (y_n)_{n\in\mathbb{N}^+} \in \tilde{X}$.

Definition 30. Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}^+}$ a sequence of elements of X. Let further $\mathbf{N} : \mathbb{R}^+ \to \mathbb{N}^+$ be a function. $(x_n)_{n\in\mathbb{N}^+}$ is a *Cauchy sequenceCauchy sequence* with modulus $\mathbf{N}$ iff for every $\varepsilon \in \mathbb{R}^+$ we have that for every $m, n \in \mathbb{N}^+$ such that $m, n \geq \mathbf{N}(\varepsilon)$ it holds that

$$d(x_m, y_n) \leq \varepsilon$$

If X is such that every Cauchy sequence converges, we call X *complete*.

Theorem 1. Let (X, d) be a metric space. $\tilde{X}$ is a complete metric space.

Proposition 5. Let (X, d) be a metric space. X is complete iff $X =_{\mathbb{V}_0} \tilde{X}$.

2.4 Total Boundedness

Total boundedness is the notion that for every $\varepsilon \in \mathbb{R}^+$ there is a subfinite set that approximates the whole metric space in the sense that every element of the metric space has a distance less than ε to some element of the subfinite set. Total boundedness is stronger than boundedness as defined in the second section and arguably the more important of the two concepts. The definitions in this chapter are adopted from Petrakis [4].

Definition 31. Let X be a set. We define the set of subfinite subsets and the set of finite subsets:

$$\mathscr{P}^{\text{subfin}}(X) := \bigcup_{n\in\mathbb{N}} \mathscr{F}(\mathbb{N}^{\leq n}, X)$$
$$\mathscr{P}^{\text{fin}}(X) := \bigcup_{n\in\mathbb{N}} \text{Emb}(\mathbb{N}^{\leq n}, X)$$

An element of $\mathscr{P}^{\text{subfin}}(X)$ or $\mathscr{P}^{\text{fin}}(X)$ is called a *subfinite subset* or *finite subset* respectively.

Remark 7. In the above definition, subfinite and finite sets are not actually sets, but rather functions. Intuitively, we identify each (sub-) finite set with its image. Then the subfiniteness property is equivalent to the more intuitive statement that a subfinite set is a subset of a finite set. In classical set theory there is no difference between finite and subfinite sets, but in constructive set theory a subset of a finite set need not be finite itself.

Definition 32. Let $(X, =_X, d)$ be a metric space and let $\mathbf{n} \in \mathbb{N}$ such that $A : \mathbb{N}^{\leq \mathbf{n}} \to X$ is a subfinite metric subset of X. Let further $\varepsilon > 0$ and let $f : X \to \mathbb{N}^{\leq \mathbf{n}}$. Then we call (A, h) a *subfinite ε-approximation* of X iff for all $x \in X$ we have that $d\,(x, A(h(x))) < \varepsilon$.

If $\boldsymbol{\alpha} : \mathbb{R}^+ \to \mathscr{P}^{\text{subfin}}(X) \times \bigcup_{n\in\mathbb{N}} \mathscr{F}(X, \mathbb{N}^{\leq \mathbf{n}})$ is a function such that for all $\varepsilon \in \mathbb{R}^+$ we have that $\boldsymbol{\alpha}_\varepsilon := (A_\varepsilon, h_\varepsilon)$ is a subfinite ε-approximation for X, then we call $(X, =_X, d)$ a *totally bounded metric space* with the *modulus of total boundedness $\boldsymbol{\alpha}$*.

Lemma 4. Let $(X, =_X, d)$ be a totally bounded metric space with modulus of total boundedness $\boldsymbol{\alpha}$. Let $\mathbf{n} \in \mathbb{N}$ such that $\boldsymbol{\alpha}_1 := (A_1, h_1)$ and $A_1 : \mathbb{N}^{\leq \mathbf{n}} \to X$ as well as $h_1 : X \to \mathbb{N}^{\leq \mathbf{n}}$. X is bounded with modulus of boundedness $\sum_{k=1}^{\mathbf{n}} d\,(A_1(0), A_1(k)) + 1$ and $A_1(0)$.

Proof. First note that in any case $0 \in \mathbb{N}^{\leq \mathbf{n}}$ and therefore X is inhabited by $A_1(0)$. Then let $x \in X$ arbitrarily. It holds that

$$
\begin{aligned}
d\,(A_1(0), x) &\leq d\,(A_1(0), A_1(h_1(x))) + d\,(A_1(h_1(x)), x) \\
&\leq \sum_{k=1}^{\mathbf{n}} d\,(A_1(0), A_1(k)) + 1
\end{aligned}
$$

therefore proving the statement. □

Proposition 6. Let $(X, =_X, d_X)$ be a totally bounded metric space with modulus of total boundedness $\boldsymbol{\alpha}$, $(Y, =_Y, d_Y)$ a metric space and $f : X \to Y$ a uniformly continuous function with modulus of uniform continuity $\boldsymbol{\omega}$. Consider the extensional subset $f(X)$ together with a modulus $\mathbf{f}^{-1}$ that assigns to every $y \in f(X)$ one preimage under f. For every $\varepsilon > 0$ let $\mathbf{n} \in \mathbb{N}$ such that for $\boldsymbol{\alpha}_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)} := (A_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)}, h_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)})$ we have that $A_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)} : \mathbb{N}^{\leq \mathbf{n}} \to X$ and $h_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon) \to X} : X \to \mathbb{N}^{\leq \mathbf{n}}$. Then define $\boldsymbol{\alpha}'_\varepsilon := (A_\varepsilon, h_\varepsilon)$ such that $A'_\varepsilon := f \circ A_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)}$

and for every $y \in Y$ it holds that $h'_\varepsilon(y) := h_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)}(\mathbf{f}^{-1}(y))$. Then $f(X)$ is totally bounded with modulus of total boundedness $\boldsymbol{\alpha}'$.

Proof. Let $y \in Y$. By construction we have that

$$d_X\left(\mathbf{f}^{-1}(y), A_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)}(h_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)}(\mathbf{f}^{-1}(y)))\right) < \boldsymbol{\omega}(\frac{1}{2}\varepsilon)$$

and since f is uniformly continuous with modulus of uniform continuity $\boldsymbol{\omega}$, we have that:

$$\begin{aligned} d_Y(y, A'_\varepsilon(h'_\varepsilon(y))) &= d_Y\left(f(\mathbf{f}^{-1}(y)), f(A_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)}(h_{\boldsymbol{\omega}(\frac{1}{2}\varepsilon)}(\mathbf{f}^{-1}(y))))\right) \\ &\leq \frac{1}{2}\varepsilon < \varepsilon \end{aligned}$$

Therefore $(A'_\varepsilon, h'_\varepsilon)$ is an ε-approximation of $f(X)$ and since ε was arbitrarily chosen, we have that $f(X)$ is totally bounded with modulus of total boundedness $\boldsymbol{\alpha}'$. □

Proposition 7. Let X be a totally bounded metric space with modulus of total boundedness $\boldsymbol{\alpha}$, and $f : X \to \mathbb{R}$ a uniformly continuous function with modulus of uniform continuity $\boldsymbol{\omega}$. Then $\sup f(X)$ and $\inf f(X)$ exist.

Proof. By proposition 6 $f(X)$ is totally bounded with modulus of total boundedness $\boldsymbol{\alpha}'$. By lemma 4 $f(X)$ is bounded with modulus of boundedness M and $A'_1(0)$. We prove the statement by utilizing the constructive least-upper-bound-property of proposition 2 and therefore we need to check the prerequisites.

Let $x, y \in \mathbb{R}$ such that $x < y$ and define $\delta := \frac{1}{4}(y - x)$. Let further $N \in \mathbb{N}$ such that $\boldsymbol{\alpha}'_\delta := (A'_\delta, h'_\delta)$ and $A'_\delta : \mathbb{N}^{\leq N} \to f(X)$ as well as $h'_\delta : f(X) \to \mathbb{N}^{\leq N}$. Then let $n \in \mathbb{N}^{\leq N}$ such that

$$A'_\delta(n) > \sup\{A'_\delta(j) \mid j \in \mathbb{N}^{\leq N}\}$$

Apparently we have that $x < x + 2\delta$ and by the constructive dichotomy of corollary 1 we have that $x < A'_\delta(n)$ or $A'_\delta(n) < x + 2\delta$. In the former case, we have the existence of some $z \in f(X)$ such that $x < z$. In the latter case, let

$z \in f(X)$ arbitrarily. We have that

$$\begin{aligned} z &\leq \underbrace{A'_\delta(h'_\delta(z))}_{<A'_\delta(n)+\delta} + \underbrace{|z - A'_\delta(h'_\delta(z))|}_{<\delta} \\ &< \underbrace{A'_\delta(n)}_{<x+2\delta} + 2\delta \\ &< x + 4\delta = y \end{aligned}$$

i. e. y is an upper bound for $f(X)$. Since all the prerequisites of 2 are met, $\sup f(X)$ exists. By considering the function $-f$ it follows that

$$\inf f(X) = -\sup -f(X)$$

exists. □

Remark 8. Note that the latter part of the proof of proposition 7 works for arbitrary totally bounded subsets of $\mathbb{R}$. Therefore every totally bounded subset (A, ι) of $\mathbb{R}$ has $\sup A$ and $\inf A$.

Proposition 8. Let $(X, =_X, d)$ be a metric space and (A, ι) a totally bounded subset. Then A is located.

Proof. Let $x \in X$. The function $d(\cdot, A) : A \to \mathbb{R}^{\geq 0}, y \mapsto d(x, \iota(y))$ is uniformly continuous. Since $(A, =_A, d_A)$ is a totally bounded metric space, by corollary 7 its infimum, i. e. $\inf\{d(x, \iota(y)) \mid y \in A\}$, exists. This holds for all $x \in X$, and therefore A is located. □

Proposition 9. Let $(X, =_X, d)$ be a totally bounded metric space with modulus of total boundedness $\boldsymbol{\alpha}$ and (Y, ι) a located subset. For $\varepsilon \in \mathbb{R}^+$ and $\mathbf{n} \in \mathbb{N}$ such that for $\boldsymbol{\alpha}_{\frac{1}{3}\varepsilon} := (A_{\frac{1}{3}\varepsilon}, h_{\frac{1}{3}\varepsilon})$ we have that $A_\varepsilon : \mathbb{N}^{\leq \mathbf{n}} \to X$ and $h_\varepsilon : X \to \mathbb{N}^{\leq \mathbf{n}}$, define $\boldsymbol{\alpha}'_\varepsilon := (A'_\varepsilon, h'_\varepsilon)$ by letting $A'_\varepsilon \in Y$ such that for every $y \in Y$ the following holds:

$$d\left(A_{\frac{1}{3}\varepsilon}(h_{\frac{1}{3}\varepsilon}(\iota(y))), \iota(A'_\varepsilon(h_{\frac{1}{3}\varepsilon}(\iota(y))))\right) < \frac{1}{3}\varepsilon + d\left(A_{\frac{1}{3}\varepsilon}(h_{\frac{1}{3}\varepsilon}(\iota(y))), Y\right)$$

Furthermore define $h'_\varepsilon := h_{\frac{1}{3}\varepsilon} \circ \iota$. Then Y is totally bounded with modulus of total boundedness $\boldsymbol{\alpha}'$.

Proof. First note that we are able to define A'_ε for every ε because Y is located and therefore the distance $d\left(A_{\frac{1}{3}\varepsilon}(h_{\frac{1}{3}\varepsilon}(\iota(y))), Y\right)$ exists which enables us to make our construction of A'_ε.

Now let $\varepsilon \in \mathbb{R}^+$ and $y \in Y$. The following holds:

$$\begin{aligned} d_Y(y, A'_\varepsilon(h'_\varepsilon(y))) &= d\left(\iota(y), \iota(A'_\varepsilon(h_{\frac{1}{3}\varepsilon}(\iota(y))))\right) \\ &\leq d\left(\iota(y), A_{\frac{1}{3}\varepsilon}(h_{\frac{1}{3}\varepsilon}(\iota(y)))\right) + d\left(A_{\frac{1}{3}\varepsilon}(h_{\frac{1}{3}\varepsilon}(\iota(y))), \iota(A'_\varepsilon(h_{\frac{1}{3}\varepsilon}(\iota(y)))\right) \\ &< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + d\left(A_{\frac{1}{3}\varepsilon}(h_{\frac{1}{3}\varepsilon}(\iota(y))), Y\right) \\ &< \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

Therefore $(A'_\varepsilon, h'_\varepsilon)$ is an ε-approximation of Y and since this holds for arbitrary ε, Y is totally bounded with modulus of total boundedness $\boldsymbol{\alpha}'$. □

2.5 Compactness

In this chapter we introduce the notion of compactness as well as related concepts. They are essential for our later discussion about local compactness.

Definition 33. A metric space (X, d) is *compact* iff it is totally bounded, i. e. it has a modulus of total boundedness $\boldsymbol{\alpha}$, and complete.

The next theorem can be understood – impredicatively – as the statement, that every compact metric space is the union of subfinitely many compact subsets with arbitrarily small diameters. In the following, we first construct such subsets and then prove the necessary properties.

Theorem 2. Let X be a compact metric space with modulus of total boundedness $\boldsymbol{\alpha}$ and $\varepsilon \in \mathbb{R}^+$. Then there exists $n \in \mathbb{N}$ and compact metric subsets $(X_0, \iota_0), \dots, (X_n, \iota_n)$ inhabited by $x_1, \dots, x_n$ respectively such that for all $j \in \mathbb{N}^{\leq n}$ we have that $(X_j, \iota_j) \subseteq [d_{x_j} < \varepsilon]$ and $X = \bigcup_{j=1}^{n} X_j$, i. e. $(X_0, \iota_0), \dots, (X_n, \iota_n)$ is an ε-covering of X.

Proof. Let $\varepsilon \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $\boldsymbol{\alpha}_{3^{-2}\varepsilon} := (A_{3^{-2}\varepsilon}, h_{3^{-2}\varepsilon})$ and $A_{3^{-2}\varepsilon} : \mathbb{N}^{\leq n} \to X$ as well as $h_{3^{-2}\varepsilon} : X \to \mathbb{N}^{\leq n}$. In the following we inductively construct for every $j \in \mathbb{N}^{\leq n}$ a sequence $(A_j^k)_{k \in \mathbb{N}}$ of subfinite subsets of X corresponding (by remark 7) to the actual subsets $((\hat{A}_j^k, \iota^X_{A_j^k}))_{k \in \mathbb{N}}$ respectively:

In the base case, we define the subfinite set $A_j^1 : \mathbb{N}^{\leq 0} \to X$ such that $A_j^1(0) := A_{3^{-2}\varepsilon}(j)$, which corresponds to the extensional subset $\hat{A}_j^1$ with

$$\hat{A}_j^1 =_{\mathcal{P}(X)} \{A_{3^{-2}\varepsilon}(j)\}$$

i. e. the sets $\hat{A}^1_j$ are exactly the singleton sets consisting of the points of the $3^{-2}\varepsilon$ approximation of X.

In the inductive case, assume the set $\hat{A}^k_j$ has already been defined for some $k \in \mathbb{N}^+$. Then first let $\boldsymbol{\alpha}_{3^{-(k+1)-1}\varepsilon} := (A_{3^{-(k+1)-1}\varepsilon}, h_{3^{-(k+1)-1}\varepsilon})$ and $N \in \mathbb{N}$ such that
$A_{3^{-(k+1)-1}\varepsilon} : \mathbb{N}^{\leq N} \to X$ and $h_{3^{-(k+1)-1}\varepsilon} : X \to \mathbb{N}^{\leq N}$. Now let S, T be two disjoint extensional subsets of $\mathbb{N}^{\leq N}$ such that $S \cup T =_{\mathcal{P}(\mathbb{N}^{\leq N})} \mathbb{N}^{\leq N}$ and the following statements hold:

(i) if $m \in S$, then $d\left(A_{3^{-(k+1)-1}}(m), X^k_j\right) < 3^{-k}\varepsilon$

(ii) if $m \in T$, then $\frac{1}{2}3^{-k}\varepsilon < d\left(A_{3^{-(k+1)-1}}(m), X^k_j\right)$

This is possible since $\mathbf{N}$ is finite and $\frac{1}{2}3^{-k}\varepsilon < 3^{-k}\varepsilon$. Finally, define A^{k+1}_j as an extension of A^k_j such that it corresponds to the extensional subset $\hat{A}^{k+1}_j$ with

$$\hat{A}^{k+1}_j =_{\mathcal{P}(X)} \hat{A}^k_j \cup \{A_{3^{-(k+1)-1}}(m) \mid m \in S\}$$

For every $j \in \mathbb{N}^{\leq n}$ and $k \in \mathbb{N}$ the following hold:

(i) $\hat{A}^k_j \subseteq \hat{A}^{k+1}_j$, i. e. the subsets form an increasing chain.

(ii) If $x \in \hat{A}^{k+1}_j$, then $d\left(x, \hat{A}^k_j\right) < 3^{-k}\varepsilon$.

(iii) For every $x \in X$, if $d\left(x, \hat{A}^k_j\right) < 3^{-k-1}\varepsilon$, then $d\left(x, \hat{A}^{k+1}_j\right) < 3^{-i-2}\varepsilon$.

Obviously (i) holds per definition and the witness function $\mathrm{id}\,|_{\hat{A}^{k+1}_j}$, since with each step elements are possibly added, but never removed. (ii) is true by choice of S in the above definition. For (iii), let $x \in X$ such that $d\left(x, \hat{A}^k_j\right) < 3^{-k-1}\varepsilon$ and let $m \in \mathbb{N}^{\leq N}$ such that $d\left(x, A_{3^{-k-2}\varepsilon}(m)\right) < 3^{-k-2}\varepsilon$. Then by triangle inequality, we have that

$$\begin{aligned} d\left(A_{3^{-k-2}\varepsilon}(m), \hat{A}^k_j\right) &\leq d\left(A_{3^{-k-2}\varepsilon}(m), x\right) + d\left(x, \hat{A}^k_j\right) \\ &\leq 3^{-k-2}\varepsilon + 3^{-k-1}\varepsilon \\ &< \frac{1}{2}3^{-k}\varepsilon \end{aligned}$$

and therefore, using S, T from the inductive definition, $m \notin T$, which by $S \cup T =_{\mathcal{P}(\mathbf{N})} \mathbf{N}$ implies $m \in S$, i. e. $A_{3^{-k-2}\varepsilon}(m) \in \hat{A}^{k+1}_j$. By definition of m we further have that

$$d\left(x, \hat{A}^{k+1}_j\right) \leq d\left(x, A_{3^{-k-2}\varepsilon}(m)\right) < 3^{-k-2}\varepsilon$$

i. e. (iii) holds.

Now define $\hat{A}_j := \bigcup_{k\in\mathbb{N}} \hat{A}_j^k$ and $X_j := \overline{\hat{A}_j}$ where $\overline{\hat{A}_j}$ denotes the closure of $\hat{A}_j$. We show that for all $j \in \mathbb{N}^{\leq n}$ the following statements hold:

(α) X_j is complete.

(β) X_j is totally bounded. Together with (α) this means that X_j is compact.

(γ) $X_j \subseteq [d_{A_{3^{-2}\varepsilon}(j)} \leq \varepsilon]$

(δ) $X =_{\mathcal{P}(X)} \bigcup_{j\in\mathbb{N}^{\leq n}} X_j$

A closed subset of a complete metric space, X_j is itself complete. To show the total boundedness of X_j, we first show the total boundedness of $\hat{A}_j$. For this, let $y \in \hat{A}_j$ and $k \in \mathbb{N}$ such that $y \in \hat{A}_j^k$. Let $m \in \mathbb{N}$ arbitrarily. By a dichotomy of $\mathbb{N}$ we have that $k \leq m$ or $m < k$. In the following we look at the different cases:

If $k \leq m$, then since $y \in \hat{A}_j^k$ and $\hat{A}_j^k \subseteq \hat{A}_j^m$ we have that $y \in \hat{A}_j^m$ and therefore $d\left(y, \hat{A}_j^m\right) = 0$. Now assume that $m < k$. Since by (ii) we have that $d\left(y, \hat{A}_j^{k-1}\right) < 3^{-k}\varepsilon$, we have that there is $y_{k-1} \in \hat{A}_j^{k-1}$ such that $d(y, y_{k-1}) < 3^{-k+1}\varepsilon$. Subsequently, we can analagously construct elements $y_{k-2} \in \hat{A}_j^{k-2}, \dots, y_m \in \hat{A}_j^m$ such that for every $i \in \mathbb{N}$ with $m+1 \leq i \leq k$ it holds that $d(y_k, y_{k-1}) < 3^{-k+1}\varepsilon$. Therefore we have that

$$\begin{aligned} d\left(y, \hat{A}_j^m\right) &\leq d(y_i, y_m) \leq \sum_{k=m+1}^{i} d(y_k, y_{k-1}) \\ &\leq \sum_{k=m+1}^{i} 3^{-k+1}\varepsilon = \sum_{k=m}^{i-1} 3^{-k}\varepsilon \\ &\leq \sum_{k=m}^{\infty} 3^{-k}\varepsilon = \frac{1}{2}3^{-m+1}\varepsilon \end{aligned}$$

Thus A_j^m is a subfinite $\frac{1}{2}3^{-m+1}\varepsilon$ approximation of $\hat{A}_j$. The total boundedness of $\hat{A}_j$ is therefore witnessed by $\boldsymbol{\alpha}^j$ defined by $\boldsymbol{\alpha}^j(\delta) := A_j^m$ where $3^{-m+1} < \delta$. The function $\bar{\boldsymbol{\alpha}}^j$, $\bar{\boldsymbol{\alpha}}^j(\delta) := \boldsymbol{\alpha}^j(\frac{1}{2}\delta)$ is a modulus of total boundedness of X_j.

For the sake of simplicity, in the following we define $x_j := \boldsymbol{\alpha}_{3^{-2}\varepsilon}$. To prove ($\gamma$), first note, that for all $k \in \mathbb{N}$ and all $y_k \in \hat{A}_j^k$ we have that $d\left(x_j, y_k\right) < 2\varepsilon \sum_{i=1}^{k-1} 3^{-i}$. We show this by induction:

The base case is obvious, since $y_1 = x_j$ and therefore $d\left(x_j, y_1\right) = 0$. Now assume $k \in \mathbb{N}$ such that for all $y_k \in \hat{A}_j^k$ the inequality holds and let $y_{k+1} \in \hat{A}_j^{k+1}$. By (ii) we know that $d\left(y_{k+1}, \hat{A}_j^k\right) < 3^{-k}\varepsilon$ and therefore we can choose $y_k \in \hat{A}_j^k$ such that $d\left(y_{k+1}, y_k\right) < 2 \cdot 3^{-k}\varepsilon$. Then it follows that

$$d\left(y_{k+1}, x_j\right) \leq d\left(y_{k+1}, y_k\right) + d\left(y_k, x_j\right) \leq 2 \cdot 3^{-k}\varepsilon + 2\varepsilon \sum_{i=1}^{k-1} 3^{-k}\varepsilon = 2\varepsilon \sum_{i=1}^{k} 3^{-k}$$

The proof now follows by induction. For any $y \in X_j$ we now have that

$$d\left(y, x_j\right) \leq 2\varepsilon \sum_{k=1}^{\infty} 3^{-k} = \varepsilon$$

i. e. $y \in [d_{x_j} \leq \varepsilon]$ and consequently $X_j \in [d_{x_j} \leq \varepsilon]$ witnessed by the identity function.

Lastly we need to prove (δ), while $X \subseteq \bigcup_{j=1}^{n} X_j$ is obviously the case. For the other inclusion, let $x \in X$ arbitrarily and $j \in \mathbf{n}$ such that

$$d\left(\boldsymbol{\alpha}_{3^{-2}\varepsilon}(j), x\right) < 3^{-2}\varepsilon$$

Note, that for every $k \in \mathbb{N}$ we have that $d\left(x, A_j^k\right) < 3^{-k+1}\varepsilon$. This is the case since in the base case

$$d\left(x, A_j^1\right) = d\left(x, \boldsymbol{\alpha}_{3^{-2}\varepsilon}(j)\right) < 3^{-2}\varepsilon$$

and in the inductive case, if $k \in \mathbb{N}$ such that the inequality holds, we have that directly by (iii), also $d\left(x, A_j^{k+1}\right) < 3^{-i+2}\varepsilon$. Since $3^{-k+1}\varepsilon$ becomes arbitrarily close to 0 for high k, we can construct a series $(y_k)_{k\in\mathbb{N}}$ with $y_k \in A_j^k$ for every $k \in \mathbb{N}$ such that $y_k \to x$, i. e. $x \in \overline{A_j} = X_j$ and subsequently $x \in \bigcup_{j\in\mathbf{n}} X_j$. □

Definition 34. Let $\alpha \in \mathbb{R}$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We call α *distinct* from $(\alpha_n)_{n\in\mathbb{N}}$, iff for every $n \in \mathbb{N}$ we have that $|\alpha - \alpha_n| > 0$.

Now let $P(\alpha)$ be a property of α. If $P(\alpha)$ holds for every $\alpha \in \mathbb{R}$ that is distinct from $(\alpha_n)_{n\in\mathbb{N}}$, we say that $P(\alpha)$ holds for *all but countably many* real numbers. In this case we call $(\alpha_n)_{n\in\mathbb{N}}$ the *excluded sequence* and every $\alpha \in \mathbb{R}$ distinct from $(\alpha_n)_{n\in\mathbb{N}}$ *admissible* for $P(\alpha)$.

Theorem 3. Let X be a compact metric space and $f : X \to \mathbb{R}$ a continuous function. For all but countably many $\alpha > \inf\{f(x) \mid x \in X\}$ the set

$$X_\alpha := \{x \in X \mid f(x) \leq \alpha\}$$

is compact.

Proof. For every $k \in \mathbb{N}$ let $N(k) \in \mathbb{N}$ such that, by means of theorem 2 there is some $N(k) \in \mathbb{N}$ such that for every $j \in \mathbb{N}^{\leq N(k)}$ there is a compact subset (X_j^k, ι_j^k) inhabited by $\tilde{x}_j^k$, such that $(X_0^k, \iota_0^k), \dots, (X_{N(k)}^k, \iota_{N(k)}^k)$ is a k^{-1} cover of X. Define further for every $k \in \mathbb{N}$ and $j \in \mathbb{N}^{\leq N(k)}$ the real numbers $c_{jk} := \inf f(X_j^k)$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence containing all the c_{jk}. This is possible, as for every $k \in \mathbb{N}$ there are only a finite number of indices j to consider.

Now let $\alpha \in \mathbb{R}$ such that $\alpha > \inf\{f(x) \mid x \in X\}$ and α is distinct from $(a_n)_{n\in\mathbb{N}}$. For every $k \in \mathbb{N}$ and $j \in \mathbb{N}^{\leq N(k)}$ we define the element x_j^k as $\tilde{x}_j^k$, except if $c_{jk} < \alpha$, then we let $x_j^k \in X_j^k$ such that $f(x_j^k) < \alpha$. We then proceed to define the set $A_\alpha := \{x_j^k \mid j \in \mathbb{N}^{\leq N(k)} \wedge c_{jk} < \alpha\}$.

For every $k \in \mathbb{N}$ we have that A_α is a subfinite k^{-1} approximation to X_α: Let $x \in X_\alpha$ and $j \in \mathbb{N}^{\leq N(k)}$ such that $x \in X_j^k$. In any case we have that $d\left(x, x_j^k\right) \leq \operatorname{diam} X_j^k < k^{-1}$, so it remains to show that in fact $x_j^k \in X_\alpha$. For this, we first have that $c_{jk} \leq f(x)$ due to the definition of c_{jk} and $f(x) \leq \alpha$ due to the fact that $x \in X_\alpha$. Since α is distinct from $(\alpha_n)_{n\in\mathbb{N}}$, i. e. every c_{jk}, it follows that $c_{jk} < \alpha$. Therefore by definition we have that $f(x_j^k) < \alpha$ and further $x_j^k \in X_\alpha$. Altogether we have that A_α is a subfinite k^{-1} approximation and thus X_α is totally bounded. X_α is complete and therefore compact, since it is closed. □

2.6 Local Compactness

Local compactness is in principle the notion of the existence of compact sets with arbitrarily large diameters. In this chapter we introduce the definition of local compactness of [4] and compare it to the one of [5]. Afterwards we define (Bishop) continuity and prove some important propositions.

Definition 35. Let (X, d) be a metric space inhabited by $x_0 \in X$. Let further $(K_n, \iota_{K_n}^X)_{n\in\mathbb{N}}$ be a sequence of compact subsets of X and $\boldsymbol{\kappa} : \mathbb{N} \to \mathbb{N}$ be a function. We say that (X, d) is *locally compact* with the *modulus of local compactness* $(x_0, (K_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa})$ iff for every $n \in \mathbb{N}$ it holds true that $[d_{x_0} \leq n] \subseteq K_{\boldsymbol{\kappa}(n)}$.

Remark 9. The definition of local compactness is equivalent to the impredicative property used by Bishop and Bridges [1], that every bounded subset is contained by a compact subset.

Remark 10. In his work, Mandelkern [5] uses a different definition of local compactness in a constructive setting that makes use of so-called *uniform neighbourhoods*. For two sets F, G, we say that G is a uniform neighbourhood of F, if for some $\varepsilon > 0$ the ε neighbourhood of F is a subset of G. He then proceeds to define term locally compact as follows:

Definition 36 (local compactness, Mandelkern [5])**.** A metric space X is called *locally compact*, iff there is some sequence $(H_k)_{k\in\mathbb{N}}$ of compact subsets of X such for every $k \in \mathbb{N}$ we have that H_{k+1} is a uniform neighbourhood of H_k as well as that $X = \bigcup_{k\in\mathbb{N}} H_k$.

Unlike definition 35 and similar to the definition of Bishop and Bridges [1], the definition of Mandelkern [5] is impredicative as it requires quantification over the powerset. Another difference is that it requires an ascending chain of compact subsets, while the modulus sequence in Definition 35 is not necessarily ordered.

While still an interesting approach, Mandelkern [5] did not offer insights into how one could develop a theory of integration in this framework and to the author's knowledge there have not yet been any other mathematicians working on this matter.

Example. $\mathbb{R}$ equipped with the standard metric d_ϵ given by the absolute value is locally compact, i. e. $(\mathbb{R}, d_\epsilon)$ is a locally compact metric space with the modulus of local compactness $(0, ([d_0 \leq n])_{n\in\mathbb{N}}, \mathrm{id})$ is a locally compact metric space.

Proposition 10. Let (X, d) be a locally compact metric space with modulus of local compactness $(x_1, (A_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa}')$. If $x_1 \in X$, then there is some $\boldsymbol{\kappa}' : \mathbb{N} \to \mathbb{N}$ such that $(x_1, (A_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa}')$ is also a modulus of local compactness for X.

Proof. We define another function $\boldsymbol{\kappa}' : \mathbb{N} \to \mathbb{N}$ as follows:

$$\boldsymbol{\kappa}'(n) := \boldsymbol{\kappa}(K_{n+d(x_0,x_1)})$$

where $\mathscr{K}\_n + d\,(x_0, x_1)$ denotes the canonical bound of $n + d\,(x_0, x_1)$. Now let $n \in \mathbb{N}$. The basic idea is that the ball $[d_{x_1} < n]$ is contained in the ball $[d_{x_0} \leq n + d\,(x_0, x_1)]$ which is itself contained in the compact set $A_{\boldsymbol{\kappa}(K_{n+d(x_0,x_1)})}$:

To prove this, let $x \in [d_{x_1} \leq n]$ and note that:

$$d(x, x_0) \leq d(x, x_1) + d(x_1, x_0) \leq n + d(x_0, x_1) < K_{n+d(x_0,x_1)}$$

i. e. $[d_{x_1} \leq n] \subseteq A_{\boldsymbol{\kappa}'(n)}$. Therefore $(x_1, (A_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa}')$ is also a modulus of local compactness for X. □

Definition 37. Let (X, d) be a locally compact metric space with modulus of local compactness $(x_0, (K_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa})$, (Y, d_Y) another metric space and $f : X \to Y$. Let further $\boldsymbol{\omega}_n : \mathbb{R}^+ \to \mathbb{R}^+$ be a function for every $n \in \mathbb{N}$. Then we say that f) is *(Bishop) continuous* with *modulus of (Bishop) continuity* $(\boldsymbol{\omega}_n)_{n\in\mathbb{N}}$ iff for all $n \in \mathbb{N}$ we have that f is uniformly continuous with modulus of uniform continuity $\boldsymbol{\omega}_n$. We also define the set $C(X, Y)$ of all (Bishop) continuous functions from X to Y. It is equipped with the equality $=_{C(X,Y)}$. We also define $C(X) := C(X, \mathbb{R})$.

Proposition 11. Let (X, d) be a metric space and (A, ι) a locally compact metric subspace with modulus of local compactness $(a_0, (K_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa})$. Then A is located.

Proof. For every $n \in \mathbb{N}$ we have that $K_{\boldsymbol{\kappa}(n)}$ is compact, i. e. totally bounded and therefore by 8 it is located. Then let $x \in X$ and $n \in \mathbb{N}$ such that $n > 2d(x, \iota(a_0))$. We show that $d(x, A)$ exists by proving that it is equal to $d\left(x, K_{\boldsymbol{\kappa}(n)}\right)$:

The inequality $d(x, A) \leq d\left(x, K_{\boldsymbol{\kappa}(n)}\right)$ is trivially valid (even without the above restriction of n), so it remains to show that $d\left(x, K_{\boldsymbol{\kappa}(n)}\right) \leq d(x, A)$. By definition of n we have that

$$d\left(x, K_{\boldsymbol{\kappa}(n)}\right) \leq d(x, \iota(a_0)) = 2d(x, \iota(a_0)) - d(x, \iota(a_0)) < n - d(x, \iota(a_0))$$

By corollary 1 (the constructive dichotomy) we have that for all $a \in A$ at least one of the following two cases holds:

(i) $d\left(x, K_{\boldsymbol{\kappa}(n)}\right) < d(x, \iota(a))$, i. e. especially $d\left(x, K_{\boldsymbol{\kappa}(n)}\right) \leq d(x, \iota(a))$ holds.

(ii) $d(x, \iota(a)) < n - d(x, \iota(a_0))$. In this case we have by triangle inequality and definition of $K_{\boldsymbol{\kappa}(n)}$ that

$$d(a, a_0) \leq d(\iota(a), x) + d(x, \iota(a_0)) < n - d(x, \iota(a_0)) + d(x, \iota(a_0)) = n$$

i. e. $a \in [d_{a_0} \leq n] \subseteq K_{\boldsymbol{\kappa}(n)}$. Therefore $d\left(x, K_{\boldsymbol{\kappa}(n)}\right) \leq d(x, \iota(a))$ as well.

Thus for all $a \in A$ $d\left(x, K_{\boldsymbol{\kappa}(n)}\right) \leq d(x, \iota(a))$ holds, which in turn implies that $d\left(x, K_{\boldsymbol{\kappa}(n)}\right) \leq d(x, A)$. □

Proposition 12. Let (X, d) be locally compact metric space with modulus of local compactness $(\tilde{x}_0, (K_n, \iota_n)_{n\in\mathbb{N}}, \tilde{\boldsymbol{\kappa}})$ and (A, ι_A^X) a closed and located subset. Then A is locally compact with some modulus of local compactness $(x_0', (K_n', \iota_n')_{n\in\mathbb{N}}, \boldsymbol{\kappa}')$.

Proof. Since A is located, it is inhabited by some $a_0 \in A$. In light of 10, there is $\boldsymbol{\kappa} : \mathbb{N} \to \mathbb{N}$ such that X is locally compact with modulus of local compactness $(x_0, (K_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa})$ where $x_0 := \iota(a_0)$. For every $\alpha \in \mathbb{R}^+$ define the extensional subset

$$X_\alpha := [d_{\iota(a_0)} \leq \alpha] = \{x \in X \mid d(x, \iota(a_0)) \leq \alpha\} \subseteq X$$

In the following we construct a sequence of sets $(K_n', \iota_n')_{n\in\mathbb{N}}$ such that A is locally compact with modulus of local compactness $(a_0, (K_n', \iota_n')_{n\in\mathbb{N}}, \mathrm{id})$.

At first let $c \in [n, n+1]$ such that X_{4c} is compact. This is possible due to the fact that $X_{4c} \subseteq K_{\boldsymbol{\kappa}(4n+4)}$, $K_{\boldsymbol{\kappa}(4n+4)}$ is a compact subset and by the use of 3. By construction we have that $[d_{\iota(a_0)} \leq n] \subseteq X_{4c}$ and therefore $[d_{a_0} \leq n] \subseteq X_{4c} \cap A$. Now define $M := \sup\{d(x, \iota(a_0)) \mid x \in X_{4c}\}$. Since $c < 2c$ we have the constructive dichotomy of corollary 1 that states that at least one of the cases $c < M$ or $M < 2c$ holds.

Case $M < 2c$: First note that, if $x \in X$ such that $d(x, \iota(a_0)) < 2c$ we have that $d(x, X_{4c} \cap A)$ exists and $d(x, X_{4c} \cap A) = d(x, A)$. This is due to the fact, that for every $a \in A$ such that $d(x, \iota(a)) < d(x, A) + (2c - d(x, \iota(a_0)))$ we have that

$$\begin{aligned} d(a, a_0) &\leq d(\iota(a), x) + d(x, \iota(a_0)) \\ &\leq d(x, A) + 2c \leq d(x, \iota(a_0)) + 2c < 4c \end{aligned} \tag{2.6.A}$$

i. e. $a \in X_{4c} \cap A$. If $M < 2c$, then consequently we have $d(x, \iota(a_0)) < 2c$ for every $x \in X_{4c}$. Therefore $d(x, X_{4c} \cap A)$ exists for every $x \in X_{4c}$ meaning $X_{4c} \cap A$ is located in X_{4c}. A located subset of the totally bounded set X_{4c}, by 9, it is also totally bounded; and a closed subset of the complete set X_{4c} it is complete as well. Together $X_{4c} \cap Y$ is compact. In this case define $K_n := X_{4c} \cap Y$.

Case $c < M$: Here we can find some $r \in (c, 2c)$ such that the set

$$V := \{x \in X \mid r \leq d(x, \iota(a_0)) \leq 4c\}$$

is compact by an argument similar to theorem 3. Then we define the set $U := V \cup (X_{4c} \cap A)$. We now show that U is located in X_{4c}. So let $x \in X_{4c}$ arbitrarily. Due to $r < 2c$ we have the dichotomy $r < d(x, \iota(a_0))$ or $d(x, \iota(a_0)) < 2c$.

If $r < d(x, \iota(a_0))$, then we have that $x \in V$ and therefore $d(x, U)$. If on the other hand $d(x, \iota(a_0)) < 2c$, then by the reasoning involving the equation 2.6.A we know that $d(x, X_{4c} \cap A)$ exists. Then

$$d(x, U) = \min\{d(x, V), d(x, X_{4c} \cap A)\}$$

due to the general fact that for any two located subsets M_0, M_1 also the subset $M_0 \cap M_1$ is located with $d(x, M_0 \cap M_1) = \min\{d(x, M_1), d(x, M_2)\}$. Therefore U is located in X_{4c} and again by 9 totally bounded. Then $\overline{U}$ is also totally bounded and additionally it is complete and together compact. In this case define $K_n := \overline{U}$.

Concluding, by our definitions, A is locally compact with modulus of local compactness $(x_0', (K_n', \iota_n')_{n \in \mathbb{N}}, \boldsymbol{\kappa}')$ as for every $n \in \mathbb{N}$ it holds true that $[d_{a_0} \leq n] \subseteq K_n$ and each of the K_n are compact sets. □

3 Integration Theory

In this chapter we discuss the theory of integration spaces presented in chapter 6, section 1 of Bishop and Bridges [1]. While in Classical Mathematics one uses primarily total functions in the context of integration, the approach of Bishop and Bridges [1] is to utilize partial functions instead. However, the definition of an integration space of Bishop and Bridges [1] makes use of a "subset of $\mathfrak{F}(X)$", i. e. a subset of the set of partial functions. Since the membership condition of $\mathfrak{F}(X)$ requires quantification over the universe $\mathbb{V}_0$, it is a class and as such the original formulation of Bishop and Bridges [1] is impredicative. We use our previously made definitions as well as an amendment to the definition of an integration space as presented in Petrakis [4] in order to remove this impredicativity. Again, the chapter follows the general outline and proofs of the respective chapter of Bishop and Bridges [1].

3.1 Functions with Compact Support

The continuous functions with compact support play an important role in the construction of our integration theory. In this section we first define the set $C^{\text{supp}}(X)$ and also the corresponding set $\text{Supp}(X)$ of partial functions.

Definition 38. Let (X, d) be a metric space, (S, ι_S^X) a located metric subset of X and $f : X \to \mathbb{R}$. S is called a *support* of f, iff for all $x \in X$ such that $d\,(x, S) > 0$ we have that $f(x) = 0$.

Let (X, d) be a locally compact metric space with modulus of local compactness $(x_0, (K_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa})$, $f \in C(X)$ and $\mathbf{n} \in \mathbb{N}$. If $\mathbf{n}$ is such that $K_{\mathbf{n}}$ is a support of f, we call f a *function with compact support* with *modulus of compact support* $\mathbf{n}$. We then define the set $C^{\text{supp}}(X)$ of exactly the Bishop continuous functions with compact support and equip it with the equality $=_{C^{\text{supp}}(X)}$ derived from the equality $=_{\mathcal{F}(X)}$. We call the elements of $C^{\text{supp}}(X)$ *test functions*.

Proposition 13. Let (X, d) be a locally compact metric space with modulus of local compactness $(x_0, (A_n, \iota_{A_n})_{n\in\mathbb{N}}, \boldsymbol{\kappa})$, (K, ι) a compact subset of X and

$\varepsilon \in \mathbb{R}^+$. Then there is $N \in \mathbb{N}$ and for every $j \in \mathbb{N}^{\leq N}$ a non-negative test function $f_j \in C^{\text{supp}}(X)$ and a compact subset (K_j, ι_j) with $\operatorname{diam}(K_j) < \varepsilon$ that is a support for f_j such that $\sum_{k=0}^{N} f_k \leq 1$ and $\sum_{k=0}^{N} f_k(\iota_k(x)) = 1$ for all $x \in K$.

Proof. Since K is a compact set, Theorem 2 provides us with $N \in \mathbb{N}$ and for every $j \in \mathbb{N}^{\leq N}$ a compact subset (K_j, ι_j) inhabited by some y_j with $\operatorname{diam}(K_j) < \frac{\varepsilon}{2}$ such that $(K_j)_{j \in \mathbb{N}^{\leq N}}$ is a subfinite cover of K. Then for every $j \in \mathbb{N}^{\leq N}$ by Theorem 3 let $\alpha \in (0, \frac{\varepsilon}{4})$ such that $S_j := [d_{K_j} \leq \alpha]$ is a compact subset of X and define the test function $g_j \in C^{\text{supp}}(X)$ by letting $g_j(x) := \max\{0, 1 - \alpha^{-1} d(x, K_j)\}$.

If for $x \in X$ we have that $g_j(x) > 0$, then it holds that

$$g_j(x) = 1 - \alpha^{-1} d(x, K_j) > 0$$

or equivalently that $d(x, K_j) < \alpha$ and therefore $x \in S_j$, i. e. S_j is a compact support of f_j. For every $x \in S_j$ we have that

$$d(x, x_j) \leq \sup\{d(y, x_j) \mid y \in K_j\} + d(x, K_j) \leq \frac{\varepsilon}{4} + \alpha < \frac{\varepsilon}{2}$$

so every S_j has a diameter of strictly less than ε. For every $j \in \mathbb{N}^{\leq N}$ we have that S_j is also a compact support for the functions f_j defined by:

$$f_j := \frac{g_j}{\max(1, \sum_{k=1}^{n} g_k)}$$

Therefore f is a test function with modulus of compact support $K_{d(x_0, y_j) + \frac{\varepsilon}{2}}$.

It remains to show that the functions $f_0, \dots, f_n$ have the desired properties. For this let $x \in X$. Then

$$\sum_{k=0}^{N} f_k(x) = \frac{\sum_{k=0}^{N} g_k(x)}{\max(1, \sum_{k=0}^{N} g_k(x))} \leq 1$$

If $x \in K_j$ for some $j \in \mathbb{N}^{\leq N}$, then $\sum_{k=0}^{N} g_k(x) \geq 1$, i. e. we have that $\max(1, \sum_{k=0}^{N} g_k(x)) = \sum_{k=0}^{N} g_k(x)$, and therefore

$$\sum_{k=0}^{N} f_k(x) = \frac{\sum_{k=0}^{N} g_k(x)}{\max(1, \sum_{k=0}^{N} g_k(x))} = \frac{\sum_{k=0}^{N} g_k(x)}{\sum_{k=0}^{N} g_k(x)} = 1$$

□

Definition 39. Let (X, d) be a locally compact metric space with modulus of local compactness $(x_0, (K_n, \iota_n)_{n \in \mathbb{N}}, \boldsymbol{\kappa})$. In analogy to proposition 1 we define the $C^{\text{supp}}(X)$-set of partial functions from X to $\mathbb{R}$

$$\text{Supp}(X) := (\lambda_0, \mathscr{E}^{C^{\text{supp}}(X)}, \lambda_1, \mathscr{P}^{\mathbb{R}})$$

from the set $C^{\text{supp}}(X)$ in the following way:

(i) $\lambda_0 : C^{\text{supp}}(X) \rightsquigarrow \mathbb{V}_0$ is the constant non-dependent assignment routine given by $\lambda_0(f) := X$ for all $f \in C^{\text{supp}}(X)$

(ii) $\mathscr{E}^{C^{\text{supp}}(X)} : \curlywedge_{i \in I} \mathscr{F}(X, X)$ is the constant dependent operation defined by $\mathscr{E}^{C^{\text{supp}}(X)}(f) := \text{id}_X$ for all $f \in C^{\text{supp}}(X)$

(iii) $\lambda_1 : \curlywedge_{(f,g) \in D(C^{\text{supp}}(X))} \mathscr{F}(X, X)$ is the constant dependent operation given by $\lambda_1(f, g) := \text{id}_X$ for all $f, g \in C^{\text{supp}}(X)$ with $f =_{C^{\text{supp}}(X)} g$

(iv) $\mathscr{P}^{\mathbb{R}} : \curlywedge_{f \in C^{\text{supp}}(X)}$ is the dependent operation defined by $\mathscr{P}^{\mathbb{R}}(f) := f$ for every $f \in C^{\text{supp}}(X)$.

3.2 The Integration Space of a Locally Compact Metric Space

In the following section we define the notion of an integration space using the definition of Petrakis [4] with minor changes regarding the use of moduli. An integration space consists of a locally compact metric space, an indexed set of partial functions and a function $\int$ which we call the *integral*. Due to the nature of sets of partial functions, it is much more convenient to define the integral on the index set rather than the actual set of partial functions. For this reason, Petrakis [4] even calls the resulting structure a "pre-integration space" rather than an "integration space". Note however, that this definition can easily be pushed back onto the set of partial functions itself, albeit requiring rather cumbersome notation. To simplify notation further, we also define some of the most important arithmetic operations directly on the index set in the obvious way. After the definition of an integration space, we introduce the definition of a positive measure and after proving some useful statements, we state and prove the main result of this thesis, the fact that Supp(X) together with any positive measure constitutes an integration space.

Definition 40. Let X be a set. We define some arithmetic operations on $\mathfrak{F}(X)$. Therefore let $\tilde{f} := (A_f, \iota_f, f)$ and $\tilde{g} := (A_g, \iota_g, f)$ be partial functions from X to $\mathbb{R}$. We define:

$$\tilde{f} \bigcirc \tilde{g} := (A_f \cap A_g, \iota^X_{A_f \cap A_g}, f \bigcirc g)$$

where $\bigcirc$ is one of $+, -, \cdot$ and $\wedge$. We also define the absolute value as well as the scalar multiplication with some $\alpha \in \mathbb{R}$ as follows:

$$|\tilde{f}| := (A_f, \iota_f, |f|)$$
$$\alpha \cdot \tilde{f} := (A_f, \iota_f, \alpha \cdot f)$$

Definition 41. Let X and I be sets, and let $(\lambda_0, \mathscr{E}, \lambda_1, \mathscr{P})$ be an I-set of partial functions from X to $\mathbb{R}$. We now use definition 40 to define arithmetic operations on the index set I. For $i, j, k \in I$ such that

$$(\lambda_0(i), \mathscr{E}_i, \mathscr{P}_i) \bigcirc (\lambda_0(j), \mathscr{E}_j, \mathscr{P}_j) =_{\mathfrak{F}(X)} (\lambda_0(k), \mathscr{E}_k, \mathscr{P}_k)$$

we define $i \bigcirc j := k$, where $\bigcirc$ is one of $+, -, \cdot$ and $\wedge$. Additionally, if k is such that

$$|(\lambda_0(i), \mathscr{E}_i, \mathscr{P}_i)| =_{\mathfrak{F}(X)} (\lambda_0(k), \mathscr{E}_k, \mathscr{P}_k)$$

then we define $|i| := k$. If $\alpha \in \mathbb{R}$ and k is such that

$$\alpha \cdot (\lambda_0(i), \mathscr{E}_i, \mathscr{P}_i) =_{\mathfrak{F}(X)} (\lambda_0(k), \mathscr{E}_k, \mathscr{P}_k)$$

we define $\alpha \cdot i := k$.

Definition 42. Let (X, d) be a locally compact metric space, $(I, =_I)$ a set, $L := (\lambda_0, \mathscr{E}, \lambda_1, \mathscr{P})$ an I-set of partial functions from X to $\mathbb{R}$ and $\int : I \to \mathbb{R}$ a function. Consider additionally the function $\mathbf{c} : I \times \mathscr{F}(\mathbb{N}, I) \to X$ and $\mathbf{p} \in I$. Then we call $(X, I, L, \int)$ an *integration space* with the modulus $(\mathbf{c}, \mathbf{p})$ iff the following properties hold true:

(i) For all $i, j \in I$ and $\alpha, \beta \in \mathbb{R}$, there exists $k \in I$ such that $\alpha \cdot i + \beta \cdot j =_I k$ and $\int k = \alpha \int i + \beta \int j$. Also there exists $l \in I$ such that $|i| =_I l$ as well as $m \in I$ such that $f \wedge 1 =_I m$, where 1 denotes the constant function 1.

(ii) If for $i \in I$ and a sequence $(i_n)_{n \in \mathbb{N}}$ of elements of I the following statement is holds: If for all $n \in \mathbb{N}$ and all $x \in \lambda_0(i_n)$ where $\mathscr{P}_{i_n}(x)$ is non-negative and it holds that $\sum_{n \in \mathbb{N}} \int i_n$ converges as well as $\sum_{n \in \mathbb{N}} \int i_n < \int i$, then

$\sum_{n\in\mathbb{N}} \mathscr{P}_{i_n}(\mathbf{c}(i,(i_n)_{n\in\mathbb{N}}))$ converges and

$$\sum_{n\in\mathbb{N}} \mathscr{P}_{i_n}(\mathbf{c}(i,(i_n)_{n\in\mathbb{N}})) < \mathscr{P}_i(\mathbf{c}(i,(i_n)_{n\in\mathbb{N}}))$$

(iii) $\int \mathbf{p} = 1$

(iv) For $i \in I$ and $m \in \mathbb{N}$ there exist $j, k \in I$ such that $i \wedge m =_I j$ as well as $|i| \wedge m^{-1} =_I k$, where m and m^{-1} denote the respective constant functions and it holds true that $\lim_{n\to\infty} \int (i \wedge n) = \int i$ as well as $\lim_{n\to\infty} \int (|i| \wedge n^{-1}) = 0$.

Remark 11. Beside the use of total functions, definition 42 differs from Bishop and Bridges [1] in that we explicitly use moduli both for the point of X as well as the function with integral 1.

Definition 43. Let X be a locally compact metric space, $\mu : C^{\text{supp}}(X) \to \mathbb{R}$ a linear map and $\mathbf{u} \in C^{\text{supp}}(X)$. We call μ a *positive measure* with modulus $\mathbf{u}$ iff $\mu(\mathbf{u}) = 1$ and for all non-negative $f \in C^{\text{supp}}(X)$ we have that $\mu(f) \geq 0$.

The above definition is equivalent to the definition of Bishop and Bridges [1], where instead the existence of some $f \in C^{\text{supp}}(X)$ such that $\mu(f) > 0$ is required. To arrive at definition 43, we merely need to define $\mathbf{u} := \frac{f}{\mu(f)}$. By linearity it follows that $\mu(\mathbf{u}) = 1$.

Lemma 5. Let (X, d) be a locally compact metric space with modulus of local compactness $(x_0, (A_n, \iota_{A_n})_{n\in\mathbb{N}}, \boldsymbol{\kappa})$, μ a positive measure on X, f a test function with modulus of compact support $\mathbf{n}$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of non-negative test functions such that $\sum_{n\in\mathbb{N}} \int f_n d\mu$ converges and

$$\sum_{n\in\mathbb{N}} \int f_n d\mu < \int f d\mu$$

If $\varepsilon \in \mathbb{R}^+$, then there is a nonnegative test function g and a compact subset K of X with $\operatorname{diam}(K) < \varepsilon$ that is a support for g, $\sum_{n\in\mathbb{N}} \int f_n g d\mu$ converges and $\sum_{n\in\mathbb{N}} \int f_n g d\mu < \int f g d\mu$.

Proof. Let $\varepsilon \in \mathbb{R}^+$. By proposition 13 there is $N \in \mathbb{N}$ and for every $j \in \mathbb{N}^{\leq N}$ a non-negative test function $g_j \in C^{\text{supp}}(X)$ as well as a compact subset (K_j, ι_j) of X with $\operatorname{diam}(K_j) < \varepsilon$ that is a support for g_j such that

$$\sum_{k=0}^{N} g_k \leq 1 \tag{3.2.A}$$

$$\sum_{k=0}^{N} g_k(\iota_{A_{\mathbf{n}}}(x)) = 1 \tag{3.2.B}$$

for every $x \in A_{\mathbf{n}}$. Due to the convergence of $\sum_{n\in\mathbb{N}} \int f_n d\mu$, by 3.2.B we have that $\sum_{j=0}^{N} \sum_{n\in\mathbb{N}} \int f_n g_j d\mu$ converges as well and

$$\sum_{j=0}^{N} \sum_{n\in\mathbb{N}} \int f_n g_j d\mu \overset{(3.2.B)}{\leq} \sum_{n\in\mathbb{N}} \int f_n d\mu < \int f d\mu \overset{(3.2.B)}{=} \sum_{j=0}^{N} \int f g_j d\mu$$

Therefore there is some $j \in \mathbb{N}^{\leq N}$ such that $\sum_{n\in\mathbb{N}} \int f_n g_j d\mu$ converges and

$$\sum_{n\in\mathbb{N}} \int f_n g_j d\mu < \int f g_j d\mu$$

Finally, letting $g := g_k$ completes the proof. □

Lemma 6. Let (X, d) be a locally compact metric space with modulus of local compactness $(x_0, (K_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa})$, $f \in C^{\text{supp}}(X)$ a test function with modulus of compact support $\mathbf{n} \in \mathbb{N}$. If $\int f d\mu > 0$, then there is $x \in K$ such that $f(x) > 0$. Since by

Proof. Let $M := \sup\{|f(x)| \mid x \in X\}$. We instead prove statement that $M > 0$. For this, we first define the auxiliary function g for every $x \in X$ as

$$g(x) := \max(1 - d(x, K_{\mathbf{n}}), 0)$$

Note first that $M \cdot g - f \geq 0$. We know that due to the dichotomy, at least one of $f(x) = 0$ or $x \in K_{\mathbf{n}}$ is true. In the first case, we have that

$$\begin{aligned} M \cdot g(x) - f(x) &= M \cdot \max(1 - d(x, K_{\mathbf{n}}), 0) - \underbrace{f(x)}_{=0} \\ &= M \cdot \max(1 - d(x, K_{\mathbf{n}}), 0) \geq 0 \end{aligned}$$

and in the second case we have that

$$M \cdot g(x) - f(x) = M \cdot \max(1 - \underbrace{d(x, K_{\mathbf{n}})}_{=0}, 0) - f(x) = M - f(x) \geq 0$$

Due to the monotonicity of the integral, we now have that

$$0 \leq \int (M \cdot g - f) d\mu = M \int g d\mu - \int f d\mu$$

which is equivalent to

$$\int f d\mu \leq M \int g d\mu$$

Since both $\int f d\mu > 0$ by assumption and $\int g d\mu \geq 0$, it follows that $M > 0$ as required. This gives us some $x \in X$ such that $f(x) > 0$. $\square$

Lemma 7. Let (X, d) be a locally compact metric space with modulus of local compactness $(x_0, (A_n, \iota_{A_n})_{n\in\mathbb{N}}, \boldsymbol{\kappa})$, $f \in C^{\text{supp}}(X)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $C^{\text{supp}}(X)$ such that $f_n \geq 0$ as well as $\sum_{n\in\mathbb{N}} \int f_n d\mu$ exists and

$$\sum_{n\in\mathbb{N}} \int f_n d\mu < \int f d\mu.$$

Then we can construct $x \in X$ such that for all $m \in \mathbb{N}$ we have that

$$\sum_{n=1}^{m} f_n(x) \leq f(x).$$

Proof. By iterated application of Lemma 5 we define the sequence $(g_n)_{n\in\mathbb{N}}$ in $C^{\text{supp}}(X)$ recursively as follows:

(a) $g_0 := f$

(b) Let $m \in \mathbb{N}$ and assume $g_k \in C^{\text{supp}}(X)$ has already been defined for all $k \in \mathbb{N}, k < m$. Then by Lemma 5 we can construct $g_m \in C^{\text{supp}}(X)$ such that g_m has a compact support K_m where $\operatorname{diam}(K_m) < m^{-1}$ and

$$\sum_{n\in\mathbb{N}} \int f_n g_1 \cdots g_m d\mu < \int f g_1 \cdots g_m d\mu$$

Cutting off the outer series, for every $m \in \mathbb{N}$ we have that

$$\sum_{n=1}^{m} \int f_n g_1 \cdots g_m < \int f g_1 \cdots g_m d\mu$$

and applying Lemma 6 on the function $f g_1 \cdots g_m - \sum_{n=1}^{m} f_n g_1 \cdots g_m$ yields some $x_m \in X$ such that

$$\Big[0 \leq \Big] \sum_{n=1}^{m} (f_n g_1 \cdots g_m)(x_m) < (f g_1 \cdots g_m)(x_m)$$

Because of $(f g_1 \cdots g_m)(x_m) > 0$ for every $m \in \mathbb{N}$, it holds that for all $k \in \mathbb{N}, k \leq m$ we have $g_k(x_m) > 0$ and therefore by induction it follows that $x_m \in K_k$. Since $\operatorname{diam}(K_k) < k^{-1}$ for all $k \in \mathbb{N}$ this means that specifically $d\,(x_m, x_k) < k^{-1}$, i. e. $(x_m)_m$ is a Cauchy sequence in X. Due to the completeness of X, $(x_m)_m$ converges to some single point $x \in X$.

Due to the fact that for all $m \in \mathbb{N}$ we have that $g_m(x_m) > 0$, it follows that $\sum_{n=1}^{m} f_n(x_m) < f(x_m)$, i. e.

$$\sum_{n=1}^{m} f_n(x) \leq f(x)$$

□

Theorem 4. Let (X, d) be a locally compact metric space with modulus of local compactness $(x_0, (K_n, \iota_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa})$ and μ a positive measure on X with modulus $\mathbf{u}$. Then there exists a function $\mathbf{c} : I \times \mathscr{F}(\mathbb{N}, C^{\text{supp}}(X)) \to X$ such that $(X, C^{\text{supp}}(X), \operatorname{Supp}(X), \mu)$ is an integration space with modulus $(\mathbf{c}, \mathbf{u})$.

Remark 12. Theorem 4 is the main result of this thesis. It states that the set $\operatorname{Supp}(X)$ of the partial functions indexed by the test functions constitutes an integration space in a sensible manner.

Proof. First we note that all the objects fulfill the respective required signatures. We therefore need to show points (i)-(iv) of definition 42.

(i): Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^{\text{supp}}(X)$ with moduli of compact support $\mathbf{n}_f, \mathbf{n}_g \in \mathbb{N}$ respectively. Then $\max(\mathbf{n}_f, \mathbf{n}_g)$ is a modulus of compact support for the function $\alpha f + \beta g$, i.e. $\alpha f + \beta g \in C^{\text{supp}}(X)$. By definition of a positive measure, μ is linear, i.e. $\int \alpha f + \beta g d\mu = \alpha \int f d\mu + \beta \int g d\mu$. Additionally, $\mathbf{n}_f$ is also the modulus of compact support for the functions $|f|$ as well as $f \wedge 1$, i.e. $|f|, f \wedge 1 \in C^{\text{supp}}(X)$.

(iii): By definition we have that $\int \mathbf{u} d\mu = 1$.

(iv): Let $f \in C^{\text{supp}}(X)$ with modulus of compact support $\mathbf{n}$. For every $r \in \mathbb{R}^+$ we have that $\mathbf{n}$ is also a modulus of compact support for $f \wedge r$. Restricting f to the compact set $K_{\mathbf{n}}$ allows us to use corollary 7 that says that

$$\sup f \mid_{K_{\mathbf{n}}} := \sup\{f(\iota_{\mathbf{n}}(x)) \mid x \in K_{\mathbf{n}}\}$$

exists. Since, if $f(x) > 0$ for some $x \in X$, then $k \in K_{\mathbf{n}}$ such that $x = \iota_{K_{\mathbf{n}}}$, we have that $\sup\{f(x) \mid x \in X\} = \sup f \mid_{K_{\mathbf{n}}}$. Now consider $k := \mathscr{K}\_\sup\{f(x) \mid x \in X\}$, i.e. the canonical bound of $\sup\{f(x) \mid x \in X\}$. Then we have for all $m \in \mathbb{N}$ such that $m \geq k$ that $f \wedge m =_{\mathcal{F}(X)} f$, i.e. the sequence $(\int f \wedge m d\mu)_{m \in \mathbb{N}}$ becomes constant and therefore $\lim_{m \to \infty} \int f \wedge m d\mu = \int f d\mu$.

For the other part, let $m \in \mathbb{N}$ and define a function $g : X \to \mathbb{R}$ by setting $g(x) := \max(1 - d(x, K_{\mathbf{n}}), 0)$ for $x \in X$. Assume that there is $x \in X$ such that $|f(x)| \wedge m^{-1} > m^{-1} g(x)$. By observing that in any case we have that $|f(x)| \wedge m^{-1} \leq m^{-1}$, we arrive at the following inequality chain:

$$m^{-1} \geq |f(x)| \wedge m^{-1} > m^{-1}(1 - d(x, K_{\mathbf{n}}))$$

By elementary transformations it is equal to

$$0 \leq m(|f(x)| \wedge m^{-1}) - 1 < d(x, K_{\mathbf{n}})$$

and specifically $d(x, K_{\mathbf{n}}) > 0$. Due to the definition of $\mathbf{n}$ it follows that $f(x) = 0$ and therefore $|f(x)| \wedge m^{-1} = 0$ which contradicts our assumption. It follows that $|f| \wedge m^{-1} \leq m^{-1} g$. Due to μ being a positive measure it follows that

$$0 \leq \int |f| \wedge m^{-1} d\mu \leq m^{-1} \int g d\mu$$

and particulary

$$0 \leq \lim_{m\to\infty} \int |f| \wedge m^{-1} d\mu \leq \lim_{m\to\infty} m^{-1} \int g d\mu = 0$$

i. e. the second part of (iv).

(ii): Let $f \in C^{\text{supp}}(X)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $C^{\text{supp}}(X)$ such that $f_n \geq 0$ as well as $\sum_{n\in\mathbb{N}} \int f_n d\mu$ exists and

$$\sum_{n\in\mathbb{N}} \int f_n d\mu < \int f d\mu.$$

Additionally consider the previously defined function g given by

$$g(x) := \max(1 - d(x, K_{\mathbf{n}}), 0)$$

and define $\alpha := \frac{1}{2} \frac{\int f d\mu - \sum_{n\in\mathbb{N}} \int f_n d\mu}{2 + \int g d\mu}$ such that

$$\sum_{n\in\mathbb{N}} \int f_n d\mu + \alpha \cdot (2 + \int g d\mu) < \int f d\mu.$$

Then let $(N(n))_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers such that

$$\sum_{k=N(n)}^{N(n+1)} \int f_n d\mu < 2^{-2n}\alpha$$

which we can since the series converges. Finally, define a new sequence $(f'_n)_{n\in\mathbb{N}}$ as follows:

$$f'_n := \begin{cases} \alpha g & n = 0 \\ f_{n'} & n = 2n' \text{ for some } n' \in \mathbb{N}^+ \\ 2^{n'} \sum_{k=N(n')}^{N(n'+1)} f_k & n = 2n' + 1 \text{ for some } n' \in \mathbb{N}^+ \end{cases}$$

i. e. $(f'_n)_{n\in\mathbb{N}} = (\alpha g, f_0, 2^0 \sum_{k=N(0)}^{N(1)} f_k, f_1, 2^1 \sum_{k=N(1)}^{N(2)} f_k, \dots)$.

Then we have that

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \int f'_n d\mu &= \int \alpha g d\mu + \sum_{n\in\mathbb{N}} \int f_n d\mu + \sum_{n\in\mathbb{N}} \int 2^n \sum_{k=N(n)}^{N(n+1)} f_k d\mu \\
&\le \alpha \int g d\mu + \sum_{n\in\mathbb{N}} \int f_n d\mu + \sum_{n\in\mathbb{N}} 2^n \sum_{k=N(n)}^{\infty} \int f_k d\mu \\
&< \alpha \int g d\mu + \sum_{n\in\mathbb{N}} f_n d\mu + \sum_{n\in\mathbb{N}} 2^n \cdot 2^{-2n}\alpha \\
&= \alpha \int g d\mu + \sum_{n\in\mathbb{N}} \int f_n d\mu + \alpha \sum_{n\in\mathbb{N}} 2^{-n} \\
&= \alpha \int g d\mu + \sum_{n\in\mathbb{N}} \int f_n d\mu + 2\alpha \\
&= \sum_{n\in\mathbb{N}} \int f_n d\mu + \alpha \cdot (2 + \int g d\mu) \\
&< \int f d\mu
\end{aligned}$$

Now we can apply Lemma 7 and construct a point $x \in K$ such that for every $m \in \mathbb{N}$

$$\alpha g(x) + \sum_{n=0}^{m} f_n(x) + \sum_{n=0}^{m} 2^n \sum_{k=N(n)}^{N(n+1)} f_k(x) \le f(x)$$

i. e. in particular $\alpha g(x) + \sum_{n=0}^{m} f_n(x) + 2^m \sum_{k=N(m)}^{N(m+1)} f_k(x) \le f(x)$. It follows that $\sum_{k=N(m)}^{N(m+1)} f_k(x) \le 2^{-m} f(x)$, i. e. $\sum_{n\in\mathbb{N}} f_n(x)$ converges and further that $\alpha g(x) + \sum_{n\in\mathbb{N}} f_n(x) \le f(x)$. Since $x \in K$, we have that $\alpha g(x) = \alpha > 0$ and therefore

$$\sum_{n\in\mathbb{N}} f_n(x) < f(x)$$

as desired. Thus we define $\mathbf{c}(f, (f_n)_n) := x$.

Together we have that $(X, C^{\text{supp}}(X), \text{Supp}(X), \mu)$ is an integration space with modulus $(\mathbf{c}, \mathbf{u})$. □

Conclusion

Within his work, Bishop has established that developing a reasonably expansive theory of integration within the framework of constructive mathematics is entirely feasible. However, its use of impredicative statements and definitions is somewhat problematic. Now, in the course of this thesis, we have demonstrated that Bishop's theory can be amended in a way that removes this impredicativity.

Unfortunately the work of creating a constructive and predicative integration theory is not yet finished—in fact, we are merely at the beginning. Building upon the notions and propositions contained in this thesis, further research has to be conducted on how to remove impredicativity from the rest of the integration and measure theory of Bishop and Bridges [1]. This includes the introduction of the notions of detachable and complemented subsets and the application of our main theorem, that states that the test functions constitute an integration space in a sensible manner. It should also be investigated to what extent choice can be avoided through the explicit use of moduli.

Furthermore, it would be interesting to see more work done on the definition of local compactness of Mandelkern [5], particularly on the relation and differences to the original definition of Bishop and Bridges [1] and the one used in this thesis. Additional investigation should also be done into how one would be able to develop a theory of integration from this definition.

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